

Energy-stable time-stepping layers for discontinuous Galerkin numerical neural operators: an SSP Runge–Kutta block and upwind DG wave integrators

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Abstract

Unrolling a time integrator turns a discretised evolution equation into a deep network whose layers are discrete operators, as in the recent numerical neural operators of Zhang. Such a network stays stable at every depth, in the forward and in the backward pass, if each layer is non-expanding in the energy norm. Our main result is a classification of time-stepping layers built on upwind discontinuous Galerkin (DG) operators: forward Euler layers are never non-expanding for polynomial degree $r \geq 1$, an SSP Runge–Kutta block of three Euler sub-layers is non-expanding under an explicit hyperbolic step limit, and the implicit, exponential and exact-projection Krylov integrators of Hochbruck, Pažur, Schulz, Thawinan and Wieners (ZAMM, 2015) are non-expanding for every step. The stability of these integrators rests on known results: Levermore’s bound for the three-stage Runge–Kutta method, B-stability of Gauss methods, contractivity of the exponential, and the known instability of forward Euler with DG. What is new is their form for upwind DG layers: a one-step norm obstruction for forward Euler that covers advection and acoustics, a short energy identity for the Runge–Kutta bound with explicit inverse-inequality constants for the upwind/interior-penalty operator, a perturbation bound for the block with a ReLU square, and the integrators of the 2015 paper as provably non-expanding layers, the Krylov ones being state-dependent. Implicit resolvent (IMEX) layers satisfy the corresponding bounds for every step. Experiments in one space dimension confirm every bound and show that the proved Runge–Kutta step is conservative by a factor between 1.12 and 2.51. They also show a trade-off between stability and flexibility: a trained Runge–Kutta DG model with 321 parameters, a fixed mesh and a fixed encoder reaches the discretisation error of its mesh ($1.96 \pm 0.01\%$ at $\nu = 10^{-4}$ over 3 training seeds) but is less accurate than black-box baselines that learn their representation, and unconstrained learned layers can blow up over long rollouts even when their one-step error is small, while a Cayley-parametrised layer is non-expanding by construction.

Keywords: discontinuous Galerkin; upwind flux; strong stability; SSP Runge–Kutta; energy method; neural operators; Krylov methods; wave equations.

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1 Introduction

Neural operators learn maps between function spaces, for example from the initial value of a time-dependent partial differential equation (PDE) to the solution at a later time [9, 37, 41, 42]. A recent line of work builds such networks directly from numerical schemes. Zhang [62] observes that one forward Euler step of a finite element discretisation of a semilinear parabolic equation has the form of a residual layer,

$$v \mapsto L_h v + C_h \sigma(Bv), \quad (1)$$

where L_h , C_h and B are discrete operators and σ is a small ReLU network that replaces the quadratic reaction term. A network of N_t such layers is the time-stepping scheme itself, and numerical analysis bounds its error and its size [62, Thm. 1]. In this reading a time step is a layer, the weights are discrete operators, and the depth of the network is the number of time steps.

Stability is then a property of the layer. The error analysis of [62] rests on the bound $\|L_h\|_{M \rightarrow M} \leq 1 + \Delta t$ [62, Lemma 3], which holds under the explicit diffusion condition $\Delta t \lambda_{\max}(M^{-1}A) \leq 2$ (with A the stiffness matrix). This ties the depth of the network to h^{-2} . For transport-dominated problems one would like a discontinuous Galerkin (DG) discretisation in space [11, 13, 16, 27, 52] with a hyperbolic step $\Delta t \sim h$. This raises the question that this paper answers: *which time-stepping layers built from an upwind DG operator are non-expanding in the energy norm, and under which step restriction?*

The answer is organised around one structural fact: the upwind DG operator is dissipative, $(A_h v, v) \geq 0$. The layers that inherit this property are safe to stack; the ones that do not are not.

Main result. For upwind DG operators we classify time-stepping layers by whether they are non-expanding in the energy norm at every depth: forward Euler layers never are for degree $r \geq 1$ (Proposition 4.1), an SSP-RK3 block is under the explicit hyperbolic step limit of Corollary 5.4, and the implicit, exponential and exact-projection Krylov integrators of [32] are for every step (Theorem 6.1).

Established facts we build on. The stability of the integrators themselves is known. The three-stage third-order Runge–Kutta method is strongly stable for semi-negative operators under $\Delta t \|L\| \leq 1$ (Levermore, in [57, Thm. 2]), and Sun and Shu classify strong stability of linear Runge–Kutta methods [56]. For DG in space, the third-order Runge–Kutta DG method is strongly (monotonicity) stable for a sufficiently small CFL number and every degree [61], [59, Prop. 4.3], while forward Euler with DG of degree $k \geq 1$ is unstable at a fixed CFL number: for $\mathbb{P}_1$ it is L^2 -stable if and only if $\Delta t/h = O(h^{1/2})$ [8, Prop. 2.1], see also [59]. Gauss collocation methods are B-stable [5, 22], the exponential of a dissipative operator is a contraction, upwind fluxes dissipate and interior penalty is coercive [1, 16].

New in this paper.

- (i) *A one-step obstruction for forward Euler* (Proposition 4.1). If the DG space contains a state that the operator moves but does not dissipate, then $\|I - \Delta t S\|_M > 1$ for every $\Delta t > 0$. For upwind DG without diffusion this holds for every degree $r \geq 1$, for advection and for the acoustic operator of [32] alike. It is the single-step, norm-level form of the known instability above, with an explicit mechanism.
- (ii) *The RK3 block for the upwind/interior-penalty DG operator* (Section 5): a short energy identity that proves Levermore’s constant (Theorem 5.1); explicit inverse-inequality constants (Proposition 5.3) that turn it into a computable hyperbolic step limit, including diffusion and inflow boundaries (Corollary 5.4); and a perturbation bound for the block with a ReLU square (Lemma 5.5).
- (iii) *The integrators of [32] as layers* (Section 6): Gauss collocation (including the implicit mid-point rule), exponential, and polynomial and rational Krylov layers are non-expanding for every step, the Krylov ones under the exact-projection condition $H_m = V_m^\top A V_m$ (Theorem 6.1). The Krylov layers are state-dependent, and the implicit layers are dense but numerically banded.
- (iv) *Implicit layers* (Section 4.2): the IMEX resolvent layer of the same architecture satisfies Zhang’s single-step bounds for every Δt (Lemmas 4.2 and 4.3); these are elementary.

- (v) *Experiments* (Section 7), all regenerated by the code that accompanies the paper: numerical checks of every bound, constructive networks for an advection–diffusion–reaction problem, trained operators, and learned wave layers.

What the layer view adds. The bounds are statements of numerical analysis; reading them as statements about networks adds three things that the numerical-analysis papers do not address. First, a single-step bound becomes a depth-independent bound for the whole network and for its input–output Jacobian, that is, for the backward pass used in training. Second, it gives a design rule for learned layers: keep the part that carries the stability fixed, or parametrise it so that the bound holds for every value of the parameters (the trained RK3-DG closure of Section 7.4 and the Cayley layer of Section 6.3); unconstrained learned layers can be accurate for one step and still blow up (Section 7.5). Third, it makes costs visible that matter for network size but not for classical stability: the locality and parameter count of the weights (banded explicit layers against dense implicit ones) and the fact that Krylov integrators are input-adaptive layers. The layer view adds no new stability theory for the integrators themselves.

What we do not claim. We prove no new convergence result for DG schemes, and our experiments are one-dimensional. The trained Runge–Kutta DG model is *less* accurate than black-box baselines in every regime we tested (with one to three training seeds per regime); this is a trade-off between stability and flexibility, not a win: the model has a fixed mesh and encoder and a provably stable linear part, while the baselines learn their representation and have no stability guarantee. The learned wave layers are studied on one linear problem. Extensions to Maxwell’s equations, learned materials, multirate stepping, learned rational layers and non-linear fluxes are outlined in Section 9.

Outline. Section 2 reviews related work and Section 3 the setting. Section 4 treats the Euler and implicit layers, Section 5 the Runge–Kutta block and Section 6 the wave integrators. Section 7 reports the experiments, Section 8 the limitations, Section 9 outlines extensions, and Section 10 concludes.

2 Related work

Operator learning. DeepONet [42] rests on the universal approximation theorem for operators of Chen and Chen [9]; Fourier neural operators [41] and the general neural-operator framework [37] parametrise integral kernels. Error and complexity estimates are available for DeepONets [15, 38, 44] and Fourier neural operators [36]. Lanthaler et al. [39] show that operators with discontinuous outputs call for nonlinear reconstruction.

Networks built from numerical methods. ReLU networks represent continuous piecewise linear finite element functions exactly [24], approximate the square map at an exponential rate [60], and emulate high-order finite elements [48]. Least-squares ReLU networks resolve discontinuous advection–reaction solutions [6]. Operator networks with numerical bases include spectral [46] and finite element [63] constructions. Zhang’s numerical neural operators [62], our starting point, unroll a finite element time-stepping scheme into the coefficient network and derive network sizes from finite element error estimates.

Stable learned dynamics and stable neural operators. Several architectures build stability into learned dynamics, with guarantees of different kinds. Residual networks can be read as discretised ODEs whose stability follows from properties of the continuous dynamics

[21]. AntisymmetricRNN [7] uses antisymmetric weights, so that the underlying ODE has purely imaginary Jacobian eigenvalues; its authors note that the forward Euler discretisation is then unstable and add diffusion, whose effect they show empirically (the same mechanism as Proposition 4.1). Manek and Kolter [43] learn a Lyapunov function together with the dynamics and obtain global exponential stability of the continuous-time model. Recurrent equilibrium networks [53] are contracting for every value of their parameters and can satisfy prescribed Lipschitz bounds; spectral normalisation [47] bounds the Lipschitz constant of a network by 1, and Cayley-parametrised recurrent networks keep the recurrent weight orthogonal [25]. Structure-preserving networks of GENERIC type [26] build energy conservation and entropy dissipation into the learned dynamics. For neural operators used autoregressively, long-rollout stability has been pursued through training strategies and architectural choices whose effect is shown empirically [4, 45]. The guarantee in this paper is of a different kind: it is a bound in the energy norm of the PDE, inherited from the dissipativity of a discrete operator, for the discrete time step actually taken, at every depth and for the backward pass. For learned parts it holds for every value of the parameters (the Cayley layer of Section 6.3, a dissipative analogue of [25]) or for the linear part whatever closure is learned (the RK3-DG model). It does not cover learned or nonlinear spatial operators.

Stability of Runge–Kutta DG. Runge–Kutta DG methods go back to Cockburn and Shu [11–13]; the SSP property [18, 55] gives stability whenever the forward Euler step is stable in the norm of interest. Energy-method analyses of explicit Runge–Kutta schemes for semi-negative operators are due to Levy and Tadmor [40] and Tadmor [57]; Sun and Shu [56] characterise strong stability for linear Runge–Kutta methods of any order. Zhang and Shu [61] analyse the L^2 -stability of third-order Runge–Kutta DG for scalar conservation laws, and Xu, Zhang, Shu and Wang [59] give a unified L^2 -stability analysis of Runge–Kutta DG methods of any order for linear hyperbolic equations. That forward Euler with $\mathbb{P}_1$ DG needs $\Delta t/h = O(h^{1/2})$ for L^2 -stability goes back to Chavent and Cockburn [8]. Interior penalty discretisations of diffusion are analysed in [1, 16, 33].

Implicit and exponential integrators. IMEX methods treat stiff linear terms implicitly and nonlinear terms explicitly [2, 3, 14]. Exponential integrators and Krylov approximations of the matrix exponential are reviewed in [28, 29]. For upwind DG discretisations of wave equations, Hochbruck et al. [32] derive the full upwind flux for heterogeneous media, prove semi-discrete convergence, and compare explicit and implicit Runge–Kutta methods with polynomial and rational Krylov methods; Section 6 reads their integrators as network layers.

3 Setting and notation

3.1 Energy norms and layers

Let $M \in \mathbb{R}^{N \times N}$ be symmetric positive definite. We write $\langle u, v \rangle_M = v^\top M u$, $\|u\|_M = \langle u, u \rangle_M^{1/2}$, and for a matrix W

$$\|W\|_M = \sup_{u \neq 0} \frac{\|Wu\|_M}{\|u\|_M} = \|M^{1/2} W M^{-1/2}\|_2.$$

For values $g \in \mathbb{R}^Q$ at quadrature points with positive weights $\omega = \text{diag}(\omega_1, \dots, \omega_Q)$ we write $\|g\|_\omega = (g^\top \omega g)^{1/2}$ and $\|C\|_{\omega \rightarrow M} = \sup_g \|Cg\|_M / \|g\|_\omega$. All semi-discrete systems below have the form

$$Mu'(t) + Au(t) = MG(u(t)), \quad S := M^{-1}A, \quad (2)$$

with $G \equiv 0$ for the wave problems.

Definition 3.1. A is *dissipative* if $A + A^\top \geq 0$, equivalently $\langle Su, u \rangle_M \geq 0$ for all u .

In the coordinates $\tilde{u} = M^{1/2}u$ the norm $\|\cdot\|_M$ is the Euclidean norm, S becomes $\tilde{S} = M^{-1/2}AM^{-1/2}$, and $\tilde{S} + \tilde{S}^\top = M^{-1/2}(A + A^\top)M^{-1/2}$. Every statement below for a dissipative $\tilde{S}$ in the Euclidean norm is therefore a statement for S in $\|\cdot\|_M$, and $\|\Phi(-\Delta t S)\|_M = \|\Phi(-\Delta t \tilde{S})\|_2$ for every rational function Φ .

A *time-stepping layer* is the map $u^n \mapsto u^{n+1}$ of a one-step method. For linear problems it is $u^{n+1} = \Phi(-\Delta t S)u^n$, where Φ is the stability function of the method [32, eq. (20)]; we call $W = \Phi(-\Delta t S)$ the *weight* of the layer. A layer is *non-expanding* if $\|W\|_M \leq 1$. A network of N_t such layers applies W^{N_t} , and its input–output Jacobian is W^{N_t} as well, so a non-expanding layer gives $\|W^{N_t}\|_M \leq 1$ at every depth, in the forward and in the backward pass. With a reaction term the relevant requirement is $\|L_h\|_{M \rightarrow M} \leq 1 + O(\Delta t)$ for the linear part, as in [62, Lemma 3].

3.2 Model problem I: advection–diffusion–reaction

We consider

$$u_t + \beta u_x - \nu u_{xx} = u(1 - u), \quad x \in (0, 1), t \in (0, T], \quad (3)$$

with $\beta > 0$, $\nu \geq 0$, and either periodic boundary conditions or $u(0, t) = 0$ at the inflow boundary (and $u(1, t) = 0$ when $\nu > 0$). The global and mesh Péclet numbers are $\text{Pe} = \beta/\nu$ and $\text{Pe}_h = \beta h/(2\nu)$.

DG space and basis. Let $x_k = kh$, $h = 1/N_e$, $I_k = (x_k, x_{k+1})$, and let V_h be the space of functions that are polynomials of degree $\leq r$ on each I_k , of dimension $N = N_e(r + 1)$. We use the scaled Legendre basis

$$\psi_{k,\ell}(x) = \sqrt{\frac{2\ell+1}{h}} P_\ell\left(\frac{2(x-x_{k+1/2})}{h}\right) \mathbf{1}_{I_k}(x), \quad 0 \leq \ell \leq r,$$

which is L^2 -orthonormal, so that $M = I$ and $\|u\|_M = \|u_h\|_{L^2(0,1)}$ for $u_h = \sum u_i \psi_i$. For a face x_f we write $u_h(x_f^\pm)$ for the one-sided traces and $[[u_h]]_f = u_h(x_f^+) - u_h(x_f^-)$. On a boundary face, $[[u_h]]_f$ is the inner trace.

Upwind advection. The advection form is

$$a(u_h, v_h) = \sum_k \left[-\beta \int_{I_k} u_h v_h' dx + \beta u_h(x_{k+1}^-) v_h(x_{k+1}^-) - \beta \hat{u}_k v_h(x_k^+) \right],$$

with the upwind value $\hat{u}_k = u_h(x_k^-)$ at interior faces, $\hat{u}_0 = u_h(x_{N_e}^-)$ for periodic boundary conditions and $\hat{u}_0 = 0$ at the inflow boundary.

Symmetric interior penalty. The diffusion form is

$$s(u_h, v_h) = \sum_k \int_{I_k} u_h' v_h' dx - \sum_f \left(\{u_h'\}_f [[v_h]]_f + \{v_h'\}_f [[u_h]]_f \right) + \frac{\sigma}{h} \sum_f [[u_h]]_f [[v_h]]_f,$$

where $\{\cdot\}_f$ is the average of the two traces at an interior face and the inner trace at a boundary face; the sums run over all faces (including the two boundary faces in the Dirichlet case), and we use $\sigma = 2(r + 1)^2$ throughout. The matrices are $(A_{\text{adv}})_{ij} = a(\psi_j, \psi_i)$, $(A_{\text{dif}})_{ij} = s(\psi_j, \psi_i)$ and $A = A_{\text{adv}} + \nu A_{\text{dif}}$; since $M = I$, the operator of (2) is $S = A$.

Reaction and quadrature. Let x_q be n_q Gauss points per element, with weights w_q , exact for polynomials of degree $3r$, and $B_{qi} = \psi_i(x_q)$. Then $B^\top \mathcal{W} B = M = I$, and (3) becomes (2) with

$$G(u) = B^\top \mathcal{W} (Bu - (Bu)^{\odot 2}), \quad (4)$$

where $\odot^2$ is the componentwise square.

The next lemma collects the inverse and trace inequalities we need, with explicit constants. Let D_r be the differentiation matrix of the orthonormal Legendre basis $\phi_j = \sqrt{(2j+1)/2} P_j$ on $[-1, 1]$, that is $\phi'_j = \sum_i (D_r)_{ij} \phi_i$; its entries are $(D_r)_{ij} = \sqrt{(2i+1)(2j+1)}$ for $j > i$ with $j-i$ odd, and 0 otherwise. Let $\tau_r^2 = \sum_{j=0}^r \phi'_j(1)^2$.

Lemma 3.2 (trace and inverse inequalities). *For $w \in \mathbb{P}_r(I_k)$ and x an end point of I_k ,*

$$|w(x)| \leq \frac{r+1}{\sqrt{h}} \|w\|_{I_k}, \quad \|w'\|_{I_k} \leq \frac{2\|D_r\|}{h} \|w\|_{I_k}, \quad |w'(x)| \leq \frac{2\sqrt{2}\tau_r}{h^{3/2}} \|w\|_{I_k}.$$

Proof. Write $w = \sum_j c_j \psi_{k,j}$, so $\|w\|_{I_k} = |c|$. By Cauchy–Schwarz, $|w(x)|^2 \leq |c|^2 \sum_j \psi_{k,j}(x)^2$, and $\sum_j \psi_{k,j}(x)^2 = \sum_j (2j+1)/h = (r+1)^2/h$ because $P_j(\pm 1)^2 = 1$. Next, $\psi'_{k,j} = (2/h) \sum_i (D_r)_{ij} \psi_{k,i}$, so $w' = (2/h) \sum_i (D_r c)_i \psi_{k,i}$ and $\|w'\|_{I_k} = (2/h) \|D_r c\|$. Finally $\psi'_{k,j}(x)^2 = (2/h)^3 \phi'_j(\pm 1)^2$, and $\phi'_j(-1)^2 = \phi'_j(1)^2$, so $\sum_j \psi'_{k,j}(x)^2 = 8\tau_r^2/h^3$. $\square$

Lemma 3.3 (dissipativity of the DG operator). *For every $u_h \in V_h$:*

- (a) $a(u_h, u_h) = \frac{\beta}{2} \sum_f \llbracket u_h \rrbracket_f^2$, where the sum runs over the N_e faces in the periodic case and over all $N_e + 1$ faces (with the inner traces at $x = 0$ and $x = 1$) in the inflow case;
- (b) if $\sigma > 3r^2$ (and $N_e \geq 2$ in the inflow case), then

$$s(u_h, u_h) \geq \frac{1}{2} \sum_k \|u'_h\|_{I_k}^2 + \frac{\sigma - 3r^2}{h} \sum_f \llbracket u_h \rrbracket_f^2 \geq 0;$$

- (c) consequently $A + A^\top \geq 0$ for every $v \geq 0$ whenever $\sigma > 3r^2$. The choice $\sigma = 2(r+1)^2$ satisfies $\sigma > 3r^2$ for $r \leq 4$.

Proof. (a) On each element, $-\beta \int_{I_k} u_h u'_h dx = -\frac{\beta}{2} (u_h(x_{k+1}^-)^2 - u_h(x_k^+)^2)$, so the contribution of I_k is $\frac{\beta}{2} u_h(x_{k+1}^-)^2 + \frac{\beta}{2} u_h(x_k^+)^2 - \beta \hat{u}_k u_h(x_k^+)$. Collecting the terms at each face x_k with an upwind neighbour gives $\frac{\beta}{2} (u_h(x_k^+) - u_h(x_k^-))^2$. At the inflow face $\hat{u}_0 = 0$ leaves $\frac{\beta}{2} u_h(0^+)^2$, and at the outflow face only $\frac{\beta}{2} u_h(1^-)^2$ remains.

(b) Put $a_k = \|u'_h\|_{I_k}$ and $J^2 = \sum_f \llbracket u_h \rrbracket_f^2$. Since $u'_h|_{I_k} \in \mathbb{P}_{r-1}$, Lemma 3.2 (with $r-1$ in place of r) gives $|u'_h(x)| \leq rh^{-1/2} a_k$ at the end points of I_k . Hence $|\llbracket u'_h \rrbracket_f| \leq rh^{-1/2} m_f$ with $m_f = \frac{1}{2}(a_{k-1} + a_k)$ at an interior face between I_{k-1} and I_k , and $m_f = a_k$ at a boundary face of I_k . An interior face contributes $m_f^2 \leq \frac{1}{2}(a_{k-1}^2 + a_k^2)$, and for $N_e \geq 2$ every element lies on at most one boundary face, so $\sum_f m_f^2 \leq \frac{3}{2} \sum_k a_k^2$. By Cauchy–Schwarz and Young’s inequality,

$$2 \left| \sum_f \llbracket u'_h \rrbracket_f \llbracket u_h \rrbracket_f \right| \leq 2rh^{-1/2} \left(\frac{3}{2} \sum_k a_k^2 \right)^{1/2} J \leq \frac{1}{2} \sum_k a_k^2 + \frac{3r^2}{h} J^2,$$

and (b) follows. (c) follows from (a), (b) and $u^\top (A + A^\top) u = 2a(u_h, u_h) + 2vs(u_h, u_h)$. $\square$

Part (a) is the classical upwind identity, and (b) is the standard coercivity argument for the symmetric interior penalty method [1, 16]; we give it with explicit constants because Section 5 uses them.

3.3 Model problem II: linear acoustics

In one space dimension we write linear acoustics in a medium with density ρ and sound speed c as the first-order system $\rho v_t + p_x = 0$, $p_t/(\rho c^2) + v_x = 0$ on $(0, 1)$, with periodic or pressure-release ($p = 0$) boundary conditions; this is a heterogeneous form of the acoustic system of [32, Sect. 2.2]. We write r for the polynomial degree in both model problems (p in [32]); here p is the pressure. The continuous operator conserves energy [32, eq. (2)]. The upwind DG discretisation of [32, Sect. 4] uses the exact solution of the Riemann problem at each face; with $Z = \rho c$ the star states are

$$v^* = \frac{Z_L v_L + Z_R v_R + p_L - p_R}{Z_L + Z_R}, \quad p^* = \frac{Z_R p_L + Z_L p_R + Z_L Z_R (v_L - v_R)}{Z_L + Z_R}.$$

In matrix form the semi-discrete system is $Mu' + Au = 0$ [32, eq. (18)], with a block-diagonal mass matrix M , and its solution is $u(t) = \exp(-tM^{-1}A)u_0$ [32, eq. (19)]. The face-dissipation identity [32, eq. (17)] expresses $(A_h v_h, v_h)$ as a weighted sum of squared jumps, so A is dissipative in the sense of Definition 3.1; this is also the content of their assumption (16), which bounds the flux defects by $C_A(A_h v_h, v_h)$ and underlies their semi-discrete error estimate [32, Thm. 2].

Remark 3.4 (the boundary jump on pressure-release faces). *What [32] prints.* For the acoustic operator, [32, Sect. 4.3, p. 246] treats Dirichlet (pressure-release) faces by setting $[[p_h]]_{K,f} = 2p_{K,h}$ and $[[\mathbf{q}_h]]_{K,f} \cdot \mathbf{n}_{K,f} = 0$ (the line before eq. (17); there $(\mathbf{q}, p)$ are flux and pressure and c_K is a material coefficient).

What follows from the definitions there. The jump is defined as $[[\mathbf{v}_h]]_{K,f} = \mathbf{v}_{K_f,h} - \mathbf{v}_{K,h}$, with the exterior state on a boundary face chosen according to the boundary condition [32, Sect. 4.1, p. 245]. Put $[[p_h]]_{K,f} = \kappa p_{K,h}$ and $[[\mathbf{q}_h]]_{K,f} \cdot \mathbf{n}_{K,f} = 0$ on a boundary face f of K . Inserting this into the face terms of the acoustic operator on p. 246, the face f contributes

$$\left(1 + \frac{\kappa c_K}{c_K + c_{K_f}}\right) (\mathbf{q}_{K,h} \cdot \mathbf{n}_{K,f}, p_{K,h})_{0,f} - \frac{\kappa}{c_K + c_{K_f}} \|p_{K,h}\|_{0,f}^2$$

to the left-hand side of eq. (17) and $\frac{\kappa^2}{2(c_K + c_{K_f})} \|p_{K,h}\|_{0,f}^2$ to its right-hand side. These agree for all $p_{K,h}$ and $\mathbf{q}_{K,h}$ only if $\kappa = -2$ and $c_{K_f} = c_K$. For $\kappa = 2$ the contribution is $-\frac{2}{c_K + c_{K_f}} \|p_{K,h}\|_{0,f}^2$ when $\mathbf{q}_{K,h} \cdot \mathbf{n}_{K,f} = 0$, which is negative. The value $\kappa = -2$ is also what the mirror state $p_{K_f} = -p_K$ gives (the exterior state for which the Riemann solution has $p^* = 0$), and it matches the electromagnetic case on p. 247, where $\mathbf{n}_{K_f} \times \mathbf{E}_{K_f} = -\mathbf{n}_{K,f} \times \mathbf{E}_K$ yields $\mathbf{n}_{K_f} \times [[\mathbf{E}]]_{K,f} = -2\mathbf{n}_{K,f} \times \mathbf{E}_K$.

Numerical support. In our layered test problem the smallest eigenvalue of the symmetric part of A is -7.12 with $[[p_h]]_{K,f} = +2p_{K,h}$ and -4×10^{-15} with $[[p_h]]_{K,f} = -2p_{K,h}$.

The sign in the printed formula is a misprint: throughout we use $[[p_h]]_{K,f} = -2p_{K,h}$, which is the intended value and the one for which the boundary terms reproduce eq. (17) as printed; with it the analysis of [32], which rests on (17), applies unchanged. The electromagnetic case on p. 247 has the same structure; see Section 9.

As a check of our implementation, the semi-discrete error at $T = 0.5$ in a layered medium converges at the rates 2.01, 3.00, 4.00, 5.00 for $r = 1, 2, 3, 4$ on uniform meshes, that is, at rate $r + 1$. This is above the rate $r + \frac{1}{2}$ guaranteed by [32, Thm. 2] for $q = r + 1$, as is common in one dimension.

3.4 The conforming setting of Zhang

For the implicit layers we use the setting of [62, Sect. 4]: continuous piecewise linear finite elements $V_h \subset H_0^1(0, 1)$ with hat basis $\phi_1, \dots, \phi_N$, mass matrix M , stiffness matrix $A_{ij} = \nu(\phi'_j, \phi'_i)$

Table 1: Notation in the three settings. In all of them $S = M^{-1}A$, $\|u\|_M^2 = u^\top Mu$, Δt is the time step (τ in [32]) and r the polynomial degree (p in [32]).

	Problem I: advection–diffusion–reaction, DG (Section 3.2)	Problem II: acoustics, DG (Section 3.3)	Fisher–KPP, continuous $\mathbb{P}_1$ (Section 3.4)
unknowns	u_h , degree r on N_e cells, $h = 1/N_e$	(v_h, p_h) (velocity, pressure), degree r on N_e cells	v_h , N interior nodes, $h = 1/(N+1)$
mass matrix M	I (orthonormal basis)	block diagonal, weights $\rho, 1/(\rho c^2)$	$\mathbb{P}_1$ mass matrix
matrix A	$A_{\text{adv}} + \nu A_{\text{dif}}$ (upwind + interior penalty)	upwind DG operator [32, eq. (13)]	$A_{ij} = \nu(\phi'_j, \phi'_i)$
operator S	$S = A$	$M^{-1}A$	$M^{-1}A$
$A + A^\top \geq 0$ by	Lemma 3.3	[32, eq. (17)], Remark 3.4	symmetry, $A \geq 0$
nonlinear term	$G(u) = B^\top \mathcal{W}(Bu - (Bu)^{\odot 2})$	none	as in Problem I
layers studied	Euler (Section 4.1), RK3 block (Section 5)	Table 3 (Section 6)	Euler, IMEX (Section 4.2)

(called S in [62]; here $S = M^{-1}A$ as in (2)), and a quadrature rule with $B^\top \mathcal{W} B = M$, for the Fisher–KPP equation $v_t - \nu v_{xx} = v - v^2$ with homogeneous Dirichlet conditions. Zhang’s forward Euler layer is $u^{n+1} = L_h u^n + C_h (Bu^n)^{\odot 2}$ with $L_h = (1 + \Delta t)I - \Delta t M^{-1}A$ and $C_h = -\Delta t M^{-1}B^\top \mathcal{W}$ [62, eqs. (13)–(14)]. Table 1 collects the notation of the three settings.

4 Euler and implicit DG-NN layers

4.1 The forward Euler layer

For (3) with $M = I$, one forward Euler step is

$$E(v) = v + \Delta t(-Av + G(v)) = L_h v + C_h (Bv)^{\odot 2}, \quad L_h = (1 + \Delta t)I - \Delta t A, \quad C_h = -\Delta t B^\top \mathcal{W}, \quad (5)$$

which is exactly a layer (1). With an orthonormal DG basis, C_h is block diagonal (element-local) and L_h couples only neighbouring elements, so the density of M^{-1} that [62, Remark 5] points out for conforming elements does not arise.

Proposition 4.1 (forward Euler). *Let A be dissipative and $\Delta t > 0$.*

- (a) $\|I - \Delta t S\|_M \leq (1 + \Delta t^2 \|S\|_M^2)^{1/2} \leq 1 + \frac{1}{2} \Delta t^2 \|S\|_M^2$.
- (b) *If some $w \neq 0$ satisfies $\langle Sw, w \rangle_M = 0$ and $Sw \neq 0$, then $\|(I - \Delta t S)w\|_M > \|w\|_M$ and $\|(1 + \Delta t)I - \Delta t S\|_M > (1 + \Delta t)\|w\|_M$ for every $\Delta t > 0$.*
- (c) *Such a w exists for the upwind DG operator of Section 3.2 with $\nu = 0$ and $r \geq 1$ (periodic or inflow), and for the upwind DG acoustic operator of Section 3.3 with $r \geq 1$ (periodic or pressure-release).*
- (d) *For $r = 0$ and $\nu = 0$, $\|I - \Delta t A\| \leq 1$ whenever $c := \beta \Delta t / h \leq 1$. In the periodic case with $N_e \geq 2$, $\|I - \Delta t A\| > 1$ for every $c > 1$.*

Proof. Work in the coordinates of Section 3, so $\|\cdot\|_M$ is Euclidean and $S + S^\top \geq 0$. For every u ,

$$\|(I - \Delta t S)u\|^2 = \|u\|^2 - \Delta t \langle (S + S^\top)u, u \rangle + \Delta t^2 \|Su\|^2. \quad (6)$$

(a) follows since the middle term is ≤ 0 , and $\sqrt{1+x} \leq 1+x/2$. (b) For w as stated the middle term vanishes, so $\|(I - \Delta t S)w\|^2 = \|w\|^2 + \Delta t^2 \|Sw\|^2 > \|w\|^2$; likewise $\|((1 + \Delta t)I - \Delta t S)w\|^2 = (1 + \Delta t)^2 \|w\|^2 + \Delta t^2 \|Sw\|^2$.

(c) Advection: let w_h be the continuous piecewise linear hat function of an interior node; it lies in V_h because $r \geq 1$, and it vanishes at $x = 0$ and $x = 1$. All its jumps vanish, including the inflow and outflow traces, so $a(w_h, w_h) = 0$ by Lemma 3.3(a). Integrating by parts on each element and using that w_h is continuous with $\hat{w}_k = w_h(x_k)$ gives $a(w_h, v_h) = \beta \int_0^1 w_h' v_h dx$ for all $v_h \in V_h$. Since w_h' is a non-zero piecewise constant, it lies in V_h , so $Aw \neq 0$. Acoustics: take $v_h = w_h$ and $p_h = 0$. All jumps of v_h and p_h vanish (on pressure-release faces the jump of p_h is $-2p_h = 0$ and that of v_h is 0), so $(A_h u_h, u_h) = 0$ by [32, eq. (17)], while the pressure component of $A_h u_h$ is the projection of $w_h' \neq 0$.

(d) For $r = 0$, $A = (\beta/h)(I - P)$ with P the cyclic shift (periodic) or the lower shift with a zero first row (inflow). Then $I - \Delta t A = (1 - c)I + cP$ and $\|P\| \leq 1$, so $\|I - \Delta t A\| \leq |1 - c| + c = 1$ for $c \leq 1$. In the periodic case P is unitary with eigenvalues $e^{i\theta}$, $\theta = 2\pi j/N_e$, and $|(1 - c) + ce^{i\theta}|^2 = 1 + 2c(c - 1)(1 - \cos \theta) > 1$ for $c > 1$ and $\theta \neq 0$. $\square$

By (a), the Euler layer satisfies $\|L_h\| \leq 1 + \Delta t + \frac{1}{2}\Delta t^2 \|A\|^2$, which is $\leq 1 + 2\Delta t$ only if $\Delta t \leq 2/\|A\|^2 \sim h^2$: the parabolic-type restriction returns for hyperbolic problems. Part (b) shows that this is not an artefact of the proof. Non-expansion is stronger than L^2 -stability (uniform boundedness of the powers of the layer): for $\mathbb{P}_1$ the explicit $P^0 P^1$ DG scheme of [8], whose update formulas [8, p. 571] are forward Euler applied to the upwind $\mathbb{P}_1$ semi-discretisation, is L^2 -stable if and only if $\Delta t/h = O(h^{1/2})$ [8, Prop. 2.1], and [59] state its instability at fixed CFL number for every degree $k \geq 1$. Proposition 4.1 is the one-step version of this fact, with the mechanism made explicit. Section 7.2 measures the excess: $\|I - \Delta t A\| - 1 \approx \kappa_r c^2$ with $\kappa_r = 6.1, 31.7, 94.6$ for $r = 1, 2, 3$.

4.2 The implicit resolvent (IMEX) layer

In the conforming setting of Section 3.4 we treat the diffusion implicitly and the reaction explicitly [2, 14]:

$$(M + \Delta t A)u^{n+1} = (1 + \Delta t)Mu^n - \Delta t B^\top \mathcal{U}(Bu^n)^{\odot 2}. \quad (7)$$

Since $M + \Delta t A$ is symmetric positive definite for every $\Delta t > 0$, this is the layer $u^{n+1} = L^{\text{im}}u^n + C^{\text{im}}(Bu^n)^{\odot 2}$ with

$$L^{\text{im}} = (1 + \Delta t)(M + \Delta t A)^{-1}M, \quad C^{\text{im}} = -\Delta t(M + \Delta t A)^{-1}B^\top \mathcal{U}. \quad (8)$$

It has the architecture (1) with the same matrix sizes as the explicit layer; a ReLU network evaluates the linear maps exactly, so only the square needs approximation.

Lemma 4.2 (unconditional single-step bounds). *For every $h > 0$, $\nu \geq 0$ and $\Delta t > 0$: $\|L^{\text{im}}\|_{M \rightarrow M} \leq 1 + \Delta t$ and $\|C^{\text{im}}\|_{\mathcal{U} \rightarrow M} \leq \Delta t$.*

Proof. With $\tilde{A} = M^{-1/2}AM^{-1/2} \geq 0$, the matrix $M^{1/2}L^{\text{im}}M^{-1/2} = (1 + \Delta t)(I + \Delta t\tilde{A})^{-1}$ is symmetric with eigenvalues $(1 + \Delta t)/(1 + \Delta t\mu_k) \in (0, 1 + \Delta t]$, where $\mu_k \geq 0$ are the eigenvalues of $\tilde{A}$. For $g \in \mathbb{R}^Q$, $M^{1/2}C^{\text{im}}g = -\Delta t(I + \Delta t\tilde{A})^{-1}Y\mathcal{U}^{1/2}g$ with $Y = M^{-1/2}B^\top \mathcal{U}^{1/2}$. Since $YY^\top = M^{-1/2}B^\top \mathcal{U}BM^{-1/2} = I$, $\|Y\|_2 = 1$, and $\|(I + \Delta t\tilde{A})^{-1}\|_2 \leq 1$; hence $\|C^{\text{im}}g\|_M \leq \Delta t\|g\|_{\mathcal{U}}$. $\square$

For the network, the square is replaced by $R_\delta = \tilde{R}_\delta \circ \chi$, where χ clips to $[-M_R, M_R]$ and $\tilde{R}_\delta$ is Yarotsky's ReLU approximation of the square with $|\tilde{R}_\delta(z) - z^2| \leq \delta$ on $[-M_R, M_R]$ [60], [62, Lemma 1, Definition 1].

Lemma 4.3 (ReLU perturbation). *Let u^n follow (7) and $\tilde{u}^n$ the same recursion with $(\cdot)^{\odot 2}$ replaced by R_δ , with $\tilde{u}^0 = u^0$. Assume $\|Bu^n\|_\infty \leq M_\infty \leq M_R$ for all n (the analogue of [62, Assumption 3]), and put $C_{\text{lip}} = 1 + M_\infty + M_R$. Then for every $\Delta t > 0$ and $h > 0$, $e^n = \tilde{u}^n - u^n$ satisfies*

$$\|e^{n+1}\|_M \leq (1 + C_{\text{lip}}\Delta t)\|e^n\|_M + \Delta t\sqrt{|\Omega|}\delta.$$

Proof. $e^{n+1} = L^{\text{im}}e^n + C^{\text{im}}[R_\delta(B\tilde{u}^n) - \chi(B\tilde{u}^n)^{\odot 2} + \chi(B\tilde{u}^n)^{\odot 2} - (Bu^n)^{\odot 2}]$. The first bracket has $\mathcal{W}$ -norm at most $(\sum_q w_q)^{1/2}\delta = \sqrt{|\Omega|}\delta$. For the second, $\chi(Bu^n) = Bu^n$, and $|\chi(a)^2 - b^2| = |\chi(a) + b||\chi(a) - b| \leq (M_R + M_\infty)|a - b|$ when $|b| \leq M_\infty$, because χ is 1-Lipschitz and fixes b . Hence its $\mathcal{W}$ -norm is at most $(M_R + M_\infty)\|Be^n\|_{\mathcal{W}} = (M_R + M_\infty)\|e^n\|_M$, using $\|Bv\|_{\mathcal{W}}^2 = v^\top B^\top \mathcal{W} Bv = \|v\|_M^2$. Lemma 4.2 completes the proof. $\square$

Iterating, $\|e^{N_t}\|_M \leq T\sqrt{|\Omega|}\delta e^{C_{\text{lip}}T}$, for every Δt , so the parabolic condition $\Delta t \lesssim h^2$ of [62, Assumption 1] is not needed for the network part. The discretisation part is a separate matter:

Corollary 4.4 (network size, conditional). *In the one-dimensional setting of Section 3.4, with continuous elements of degree r on a uniform mesh of $(0, 1)$ ($r = 1$ there) and quadrature exact for degree $3r$, assume that the IMEX-Euler solution satisfies $\|v(T) - v_h^{N_t}\|_{L^2} \leq C(h^{r+1} + \Delta t)$ uniformly over the admissible initial data. Then for $\varepsilon \in (0, \frac{1}{2})$ the choices $h \sim \varepsilon^{1/(r+1)}$, $\Delta t \sim \varepsilon$ and $\delta \sim \varepsilon e^{-C_{\text{lip}}T}$ give a network with error $O(\varepsilon)$, $N_t = O(\varepsilon^{-1})$ layers, depth $O(\varepsilon^{-1} \log(1/\varepsilon))$ and width $O(h^{-1}) = O(\varepsilon^{-1/(r+1)})$.*

Proof. The triangle inequality splits the error into the assumed discretisation error and the network error of Lemma 4.3. Each layer adds the depth of R_δ , which is $O(\log(M_R^2/\delta))$ [60], [62, Lemma 1]; the width count is as in [62, Thm. 1]. $\square$

We state the corollary conditionally because we have not verified the assumptions under which such an estimate holds for the semilinear problem; for linear parabolic problems such estimates are classical [58].

Remark 4.5 (standard extension to d dimensions). Lemmas 4.2 and 4.3 use only $B^\top \mathcal{W} B = M$ and $A = A^\top \geq 0$, so they hold in any dimension. Under the assumptions of [62, Thm. 1] (quasi-uniform affine simplicial meshes in d dimensions, exact quadrature) and the same assumed error estimate, the argument of Corollary 4.4 gives width $O(h^{-d}) = O(\varepsilon^{-d/(r+1)})$; we state this as the standard extension and do not use it.

The same lemmas hold verbatim with the symmetric interior penalty matrix A_{dif} in place of A . The price of the implicit layer is a dense weight $(M + \Delta t A)^{-1}$: [62, Remark 5] distinguishes width from parameter count for exactly this reason, and Section 6.2 measures how quickly such weights decay away from the diagonal.

4.3 Exact ReLU pieces

Two small pieces that DG methods need are exactly ReLU-representable.

Proposition 4.6. (a) $\min\text{mod}(a, b) = \text{ReLU}(a - \text{ReLU}(a - b)) - \text{ReLU}(-a - \text{ReLU}(b - a))$ for all $a, b \in \mathbb{R}$.

(b) Let ψ be a $\mathbb{P}_0$ or $\mathbb{P}_1$ basis function on $I_k = (x_k, x_{k+1})$, extended by zero, and $0 < \epsilon < h/2$. The continuous piecewise linear function g_ϵ that vanishes outside I_k , equals ψ on $[x_k + \epsilon, x_{k+1} - \epsilon]$ and is linear on the two remaining pieces is a sum of four ReLUs, $g_\epsilon(x) = \sum_{i=1}^4 \gamma_i \text{ReLU}(x - b_i)$ with $b = (x_k, x_k + \epsilon, x_{k+1} - \epsilon, x_{k+1})$. It agrees with ψ at every Gauss point of I_k if $\epsilon < \frac{h}{2}(1 - \xi_{\max})$, where $\xi_{\max}$ is the largest Gauss node on $[-1, 1]$.

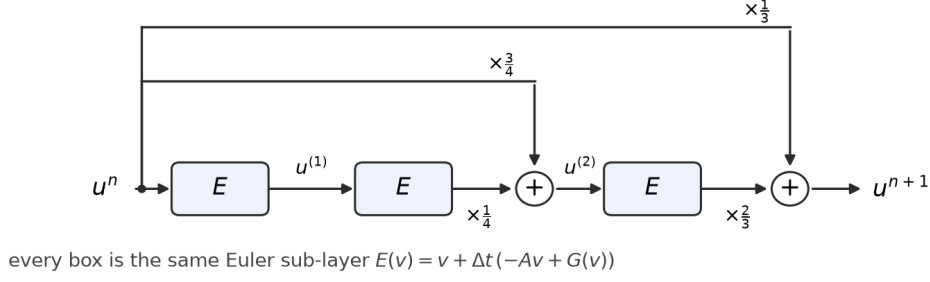


Figure 1: The SSP-RK3 DG-NN block (9): three copies of the same Euler sub-layer E and two skip connections with fixed convex weights.

Proof. (a) $a - \text{ReLU}(a - b) = \min(a, b)$ and $-a - \text{ReLU}(b - a) = -\max(a, b)$, so the right-hand side is $\max(\min(a, b), 0) - \max(-\max(a, b), 0)$, which is $\min(a, b)$ if $a, b > 0$, $\max(a, b)$ if $a, b < 0$, and 0 otherwise. (b) A continuous piecewise linear function that vanishes left of b_1 and has slope s_i on (b_i, b_{i+1}) , with $s_0 = s_4 = 0$, equals $\sum_i (s_i - s_{i-1}) \text{ReLU}(x - b_i)$. The Gauss points of I_k lie at distance $\frac{h}{2}(1 - \zeta_{\max})$ or more from the end points. $\square$

In the experiments the decoder evaluates the DG expansion directly, so (b) is not used there; it shows that the output representation of [62, Lemma 5] extends to discontinuous $\mathbb{P}_1$ functions at quadrature points.

5 The SSP-RK3 DG-NN block

5.1 The block

The three-stage SSP Runge–Kutta method of Shu and Osher [18, 55] applied to (2) with the Euler sub-layer (5) reads

$$u^{(1)} = E(u^n), \quad u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}E(u^{(1)}), \quad u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}E(u^{(2)}). \quad (9)$$

All three sub-layers share the weights L_h and C_h of (5); the convex weights are fixed numbers, not parameters (Figure 1). As a network, one carries the pair $(u^n, u^{(j)})$, so the width is $2N$ plus the width of the square sub-network, and each time step has depth three Euler sub-layers; the depth is $3N_t$. For $G \equiv 0$ the block is the matrix

$$u^{n+1} = P_3(-\Delta t S)u^n, \quad P_3(-Z) = I - Z + \frac{1}{2}Z^2 - \frac{1}{6}Z^3. \quad (10)$$

The usual SSP argument bounds the block by the norm of the Euler step [18]. Here it is not available: by Proposition 4.1 the Euler sub-layer itself is expansive for $r \geq 1$. Stability has to come from the composition.

5.2 Stability

Theorem 5.1 (SSP-RK3 block). *Let $Z \in \mathbb{R}^{N \times N}$ satisfy $H := Z + Z^\top \geq 0$. For $u \in \mathbb{R}^N$ put $u_k = Z^k u$ and $x_k = H^{1/2} u_k$. Then*

(a) *for every u ,*

$$\|P_3(-Z)u\|^2 = \|u\|^2 - \|x_0 - \frac{1}{2}x_1 + \frac{1}{6}x_2\|^2 - \frac{1}{12}\|x_1 - x_2\|^2 + \frac{1}{36}\|x_2\|^2 - \frac{1}{12}\|u_2\|^2 + \frac{1}{36}\|u_3\|^2; \quad (11)$$

(b) *if $\|H\| + \|Z\|^2 \leq 3$, in particular if $\|Z\| \leq 1$, then $\|P_3(-Z)\| \leq 1$.*

Consequently, if A is dissipative, then $\|P_3(-\Delta t S)\|_M \leq 1$ whenever $\Delta t \|S\|_M \leq 1$.

Proof. (a) Put $g_{jk} = \langle u_j, u_k \rangle$ and $s_{jk} = \langle x_j, x_k \rangle = \langle u_j, Hu_k \rangle = g_{j,k+1} + g_{j+1,k}$. Since $P_3(-Z)u = u_0 - u_1 + \frac{1}{2}u_2 - \frac{1}{6}u_3$,

$$\|P_3(-Z)u\|^2 - \|u\|^2 = -2g_{01} + g_{02} + g_{11} - \frac{1}{3}g_{03} - g_{12} + \frac{1}{3}g_{13} + \frac{1}{4}g_{22} - \frac{1}{6}g_{23} + \frac{1}{36}g_{33}.$$

Substituting $2g_{01} = s_{00}$, $g_{02} + g_{11} = s_{01}$, $g_{03} = s_{02} - g_{12}$, $g_{13} = s_{12} - g_{22}$, $2g_{23} = s_{22}$ and $2g_{12} = s_{11}$ turns the right-hand side into

$$-\left[s_{00} - s_{01} + \frac{1}{3}s_{11} + \frac{1}{3}s_{02} - \frac{1}{3}s_{12} + \frac{1}{12}s_{22}\right] - \frac{1}{12}g_{22} + \frac{1}{36}g_{33}.$$

Expanding $\|x_0 - \frac{1}{2}x_1 + \frac{1}{6}x_2\|^2 + \frac{1}{12}\|x_1 - x_2\|^2 = s_{00} - s_{01} + \frac{1}{3}s_{02} + \frac{1}{3}s_{11} - \frac{1}{3}s_{12} + \frac{1}{9}s_{22}$ shows that the bracket equals this sum minus $\frac{1}{36}s_{22}$, which is (11).

(b) Drop the two non-negative squares in (11). Since $\|x_2\|^2 = \langle u_2, Hu_2 \rangle \leq \|H\| \|u_2\|^2$ and $\|u_3\| = \|Zu_2\| \leq \|Z\| \|u_2\|$,

$$\|P_3(-Z)u\|^2 \leq \|u\|^2 + \frac{1}{36}(\|H\| + \|Z\|^2 - 3)\|u_2\|^2 \leq \|u\|^2.$$

If $\|Z\| \leq 1$ then $\|H\| \leq 2\|Z\|$ and $\|H\| + \|Z\|^2 - 3 \leq (\|Z\| - 1)(\|Z\| + 3) \leq 0$. The final statement follows with $Z = \Delta t \tilde{S}$ in the coordinates of Section 3. $\square$

Remark 5.2 (credit and sharpness). The bound $\|P_3(L)\|_H \leq 1$ for H -semi-negative L with $\Delta t \|L\|_H \leq 1$ is due to Levermore and is stated by Tadmor [57, Thm. 2], who conjectures that this constant is optimal for general semibounded, non-diagonalisable operators and notes that for skew-symmetric L the sharp condition is $\Delta t \leq \sqrt{3}/\|L\|$ [57, Remark 2]. Sun and Shu [56, Thm. 4.2] show more generally that linear p -stage Runge–Kutta methods of order p are strongly stable (for *some* step constant) when $p \equiv 3 \pmod{4}$ and are not when $p \equiv 1, 2 \pmod{4}$; forward Euler and two-stage second-order methods are in the second class. The identity (11) is a short self-contained route to the constant 1 and shows where dissipation enters: through $x_k = H^{1/2}u_k$, which vanish when Z is skew. For skew Z with eigenvalue $i\omega$ one has $|P_3(-i\omega)|^2 = 1 - \omega^4/12 + \omega^6/36$, which exceeds 1 exactly when $|\omega| > \sqrt{3}$. On our DG operators the largest admissible $\theta = \Delta t \|A\|$ lies between 1.12 and 2.51 (Section 7.2).

5.3 Explicit constants and the hyperbolic step limit

Proposition 5.3 (inverse inequalities with constants). *For the operators of Section 3.2 on a uniform mesh, with periodic or inflow boundary conditions,*

$$\|A_{\text{adv}}\| \leq C_{\text{adv}}(r) \frac{\beta}{h}, \quad \|A_{\text{dif}}\| \leq \frac{C_{\text{dif}}(r)}{h^2},$$

with

$$C_{\text{adv}} = 2\|D_r\| + 2(r+1)^2, \quad C_{\text{dif}} = 4\|D_r\|^2 + 8\sqrt{2} \tau_r(r+1) + 4\sigma(r+1)^2.$$

For $r = 0$, periodic boundary conditions and even N_e , the bound on $\|A_{\text{adv}}\|$ is attained. The constants are listed in Table 2.

Proof. We bound the bilinear forms by $C\|u_h\| \|v_h\|$. Write $a_k = \|u_h\|_{I_k}$ and $b_k = \|v_h\|_{I_k}$, and use Lemma 3.2 on each element.

Advection. The volume term satisfies $\beta \int_{I_k} u_h v_h' \leq \beta a_k \cdot \frac{2}{h} \|D_r\| b_k$; summing with Cauchy–Schwarz gives $\frac{2\beta}{h} \|D_r\| \|u_h\| \|v_h\|$. The outflow term satisfies $\beta |u_h(x_{k+1}^-) v_h(x_{k+1}^-)| \leq \beta \frac{(r+1)^2}{h} a_k b_k$, and the upwind term $\beta |\hat{u}_k v_h(x_k^+)| \leq \beta \frac{(r+1)^2}{h} a_{k-1} b_k$ (with $a_{-1} = a_{N_e-1}$ in the periodic case; the term is absent at the inflow face). Each of the two sums is at most $\beta \frac{(r+1)^2}{h} \|u_h\| \|v_h\|$.

Table 2: Constants of Proposition 5.3 ($\sigma = 2(r+1)^2$) and measured norms, largest over the configurations of Section 7.2 ($N_e \in \{8, 16, 32, 64\}$, both boundary conditions).

r	σ	$\ D_r\ $	C_{adv}	measured $\max\ A_{\text{adv}}\ h/\beta$	C_{diff}	measured $\max\ A_{\text{dif}}\ h^2$
0	2	0.000	2.00	2.00	8.0	8.0
1	8	1.732	11.46	6.00	167.7	84.0
2	18	3.873	25.75	12.18	874.3	362.1
3	32	6.522	45.04	20.95	2772.4	1058.7

Diffusion. The volume term is bounded by $\frac{4}{h^2}\|D_r\|^2\|u_h\|\|v_h\|$. At an interior face, $|\{u'_h\}| \leq \sqrt{2}\tau_r h^{-3/2}(a_{k-1} + a_k)$ and $|\llbracket v_h \rrbracket| \leq (r+1)h^{-1/2}(b_{k-1} + b_k)$; at a boundary face of I_k , $|\{u'_h\}| \leq 2\sqrt{2}\tau_r h^{-3/2}a_k$ and $|\llbracket v_h \rrbracket| \leq (r+1)h^{-1/2}b_k$. Write the face terms as $\sqrt{2}\tau_r(r+1)h^{-2}\xi_f\eta_f$ with $(\xi_f, \eta_f) = (a_{k-1}+a_k, b_{k-1}+b_k)$ at interior faces and $(\sqrt{2}a_k, \sqrt{2}b_k)$ at boundary faces. Since every element lies on two faces, $\sum_f \xi_f^2 \leq 4\sum_k a_k^2$ and $\sum_f \eta_f^2 \leq 4\sum_k b_k^2$, so each consistency term is at most $4\sqrt{2}\tau_r(r+1)h^{-2}\|u_h\|\|v_h\|$, and there are two of them. In the same way the penalty term is at most $\frac{\sigma}{h}\frac{(r+1)^2}{h} \cdot 4\|u_h\|\|v_h\|$. For $r=0$ and periodic boundary conditions $A_{\text{adv}} = (\beta/h)(I - P)$ with P the cyclic shift, and $\|I - P\| = 2$ for even N_e . $\square$

Corollary 5.4 (hyperbolic step limit). *For the DG operator $A = A_{\text{adv}} + \nu A_{\text{dif}}$, $\|P_3(-\Delta t A)\| \leq 1$ whenever*

$$\Delta t \leq \left(C_{\text{adv}}(r)\frac{\beta}{h} + C_{\text{diff}}(r)\frac{\nu}{h^2} \right)^{-1}.$$

For $\nu = 0$ this is $\Delta t \leq h/(C_{\text{adv}}(r)\beta)$, a hyperbolic condition for every degree r , in contrast to Proposition 4.1.

That the third-order Runge–Kutta DG method is strongly stable for a sufficiently small CFL number is known [61], [59, Prop. 4.3]; what Corollary 5.4 adds is an explicit constant, including diffusion and inflow boundaries. The constants are safe but not sharp (Table 2): for $r=1$ and $\nu=0$ the measured limit of the block is $c^* = 0.187$ in the variable $c = \beta\Delta t/h$, against the proved $1/C_{\text{adv}}(1) = 0.087$. In the experiments we therefore use the step $\Delta t = T/[T\|A\|_2] \leq 1/\|A\|_2$ with the computed norm, which Theorem 5.1 covers directly.

5.4 The block with a ReLU square

Lemma 5.5 (perturbation bound for the block). *Let A be dissipative with $\Delta t\|A\| \leq 1$, let u^n follow (9) and $\tilde{u}^n$ the same recursion with the square in (4) replaced by R_δ , with $\tilde{u}^0 = u^0$. Assume that all stage values of the exact trajectory satisfy $\|Bu^{(i)}\|_\infty \leq M_\infty \leq M_R$, put $C_{\text{lip}} = 1 + M_\infty + M_R$ and assume $\Delta t C_{\text{lip}} \leq 1$. Then*

$$\|e^{n+1}\| \leq (1 + 4C_{\text{lip}}\Delta t)\|e^n\| + 3\Delta t\sqrt{|\Omega|}\delta, \quad \text{hence} \quad \|e^{N_t}\| \leq 3T\sqrt{|\Omega|}\delta e^{4C_{\text{lip}}T}.$$

Proof. Write $a = \Delta t A$. Eliminating the stages in (9) gives

$$u^{n+1} = P_3(-a)u^n + \Delta t \left[\frac{1}{6}(I - a)^2 G(u^n) + \frac{1}{6}(I - a)G(u^{(1)}) + \frac{2}{3}G(u^{(2)}) \right], \quad (12)$$

and the same formula holds for $\tilde{u}$ with $\tilde{G}$ in place of G . As in the proof of Lemma 4.3, $\|\tilde{G}(v) - G(w)\| \leq C_{\text{lip}}\|v - w\| + d$ with $d = \sqrt{|\Omega|}\delta$ whenever $\|Bw\|_\infty \leq M_\infty$; here also $\|B^T \omega g\| \leq \|g\|_{\omega\omega}$ and $\|Bv\|_{\omega\omega} = \|v\|$ are used. By (6) and $\|a\| \leq 1$, $\|I - a\| \leq \sqrt{2}$. Let $e_0 = e^n$ and e_1, e_2 be the stage errors. From $e_1 = (I - a)e_0 + \Delta t(\tilde{G}(\tilde{u}^n) - G(u^n))$ and $\Delta t C_{\text{lip}} \leq 1$,

$$\|e_1\| \leq (\sqrt{2} + \Delta t C_{\text{lip}})\|e_0\| + \Delta t d \leq (1 + \sqrt{2})\|e_0\| + \Delta t d.$$

From $e_2 = \frac{3}{4}e_0 + \frac{1}{4}[(I - a)e_1 + \Delta t(\tilde{G}(\tilde{u}^{(1)}) - G(u^{(1)}))]$,

$$\|e_2\| \leq \frac{3}{4}\|e_0\| + \frac{1+\sqrt{2}}{4}\|e_1\| + \frac{1}{4}\Delta t d \leq \frac{3+\sqrt{2}}{2}\|e_0\| + \frac{2+\sqrt{2}}{4}\Delta t d,$$

using $\frac{3}{4} + \frac{(1+\sqrt{2})^2}{4} = \frac{3+\sqrt{2}}{2}$. With $\|P_3(-a)\| \leq 1$ (Theorem 5.1), $\|(I - a)^2\| \leq 2$ and (12),

$$\begin{aligned} \|e^{n+1}\| &\leq \|e_0\| + \Delta t \left[\frac{1}{3}(C_{\text{lip}}\|e_0\| + d) + \frac{\sqrt{2}}{6}(C_{\text{lip}}\|e_1\| + d) + \frac{2}{3}(C_{\text{lip}}\|e_2\| + d) \right] \\ &\leq \left(1 + \frac{10+3\sqrt{2}}{6}C_{\text{lip}}\Delta t\right)\|e_0\| + \frac{8+3\sqrt{2}}{6}\Delta t d, \end{aligned}$$

where the second step inserts the two stage bounds and uses $\Delta t C_{\text{lip}} \leq 1$ once more on the $\Delta t d$ parts: the coefficient of $C_{\text{lip}}\Delta t\|e_0\|$ is $\frac{1}{3} + \frac{\sqrt{2}}{6}(1 + \sqrt{2}) + \frac{2}{3} \cdot \frac{3+\sqrt{2}}{2} = \frac{10+3\sqrt{2}}{6}$, and that of $\Delta t d$ is at most $1 + \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} + \frac{2}{3} \cdot \frac{2+\sqrt{2}}{4} = \frac{8+3\sqrt{2}}{6}$. Since $\frac{10+3\sqrt{2}}{6} < 2.38 < 4$ and $\frac{8+3\sqrt{2}}{6} < 2.05 < 3$, this implies the claim (the decimals are rounded up). Iterating and using $(1 + 4C_{\text{lip}}\Delta t)^k \leq e^{4C_{\text{lip}}\Delta t k}$ for $k \leq N_t$ gives the second bound. $\square$

Combining Lemma 5.5 with an error estimate for Runge–Kutta DG of the form $O(h^{r+1} + \Delta t^3)$ would give a network-size statement of the type of Corollary 4.4 with $N_t = O(h^{-1})$ blocks of depth three. We do not state it: the available analyses [13, 61] are for conservation laws without the diffusion and reaction terms of (3), and we have not checked their assumptions.

Remark 5.6 (other Runge–Kutta methods). By [56, Thm. 4.2], forward Euler and two-stage second-order methods admit no step constant that makes them strongly stable for all semi-negative operators. The classical four-stage method is not strongly stable in one step, but two steps are; both results are stated in [56, Prop. 3.1 and Thm. 3.2], with credit to earlier work of the same authors. On our DG operators (Table 4) the SSP-RK2 step is non-expanding for $r = 1$ up to $c = 0.333$, but for $r = 2, 3$ only up to $c \leq 5 \times 10^{-4}$: its excess over 1 grows like θ^4 and passes our tolerance of 10^{-12} almost at once. One RK4 step behaves similarly for $r = 2, 3$ ($c \leq 4 \times 10^{-3}$), while two RK4 steps stay below 1 up to $c = 0.456, 0.164, 0.103$ for $r = 1, 2, 3$.

6 The upwind DG wave integrators of Hochbruck et al. as layers

6.1 The dictionary

For the wave problems of Section 3.3, every one-step method of [32, Sect. 5] is a layer $u^{n+1} = \Phi(-\Delta t S)u^n$ [32, eq. (20)], and the layer has no ReLU part because the problem is linear. Table 3 lists the integrators studied there, together with the RK3 block of Section 5 and forward Euler for comparison. The implicit midpoint rule $u^{n+1} = u^n - \Delta t(M + \frac{\Delta t}{2}A)^{-1}Au^n$ [32, Sect. 5] is the Cayley transform $(I + \frac{\Delta t}{2}S)^{-1}(I - \frac{\Delta t}{2}S)$; the three-stage Gauss method is implemented in [32] as a chain of resolvents with complex shifts, and its stability function is the (3,3) Padé approximation of the exponential [32, Sect. 5]. The polynomial Krylov method [32, Alg. 1, eq. (22)] computes

$$u^{n+1} = \|u^n\|_M V_m \exp(-\Delta t H_m) e_1, \quad V_m^\top M V_m = I, \quad H_m = V_m^\top A V_m, \quad (13)$$

where V_m spans the Krylov space of S and u^n ; the rational Krylov method [32, Alg. 2, eqs. (23)–(24)] has the same form with V_m spanning a rational Krylov space and $H_m = V_m^\top A V_m$ recovered from the Arnoldi quantities.

Theorem 6.1 (non-expanding wave layers). *Let A be dissipative, $S = M^{-1}A$ and $\Delta t > 0$.*

- (a) *Gauss collocation. For every $m \geq 1$ the m -stage Gauss collocation step for $u' = -Su$ is uniquely defined and satisfies $\|u^{n+1}\|_M \leq \|u^n\|_M$. In particular the implicit midpoint rule ($m = 1$) and the three-stage Gauss method of [32] are non-expanding layers for every Δt .*

Table 3: Time integrators of [32] (and two explicit methods) read as layers. “Reach” is the number of neighbouring cells that one layer couples. Bounds hold for dissipative A ; references are to this paper.

Integrator	Weight W	Reach	Bound on $\ W\ _M$
forward Euler	$I - \Delta t S$	1	> 1 for all Δt if $r \geq 1$ (Prop. 4.1)
SSP-RK3 block	$P_3(-\Delta t S)$, three sub-layers	3	≤ 1 if $\Delta t \ S\ _M \leq 1$ (Thm. 5.1)
classical RK4	degree-4 polynomial	4	no general bound [56, Prop. 3.1]
implicit midpoint	$(I + \frac{\Delta t}{2} S)^{-1} (I - \frac{\Delta t}{2} S)$	all (decaying)	≤ 1 for all Δt (Thm. 6.1)
Gauss, m stages	(m, m) Padé approximant	all (decaying)	≤ 1 for all Δt (Thm. 6.1)
exponential	$\exp(-\Delta t S)$	all (decaying)	≤ 1 for all Δt (Thm. 6.1)
Krylov (poly./rational)	state-dependent, (13), with $H_m = V_m^\top A V_m$	state- dependent	≤ 1 for all $\Delta t, m$ if $H_m = V_m^\top A V_m$ exactly (Thm. 6.1); not shown for [32, eq. (25)]

(b) Exponential. $\|\exp(-\Delta t S)\|_M \leq 1$.

(c) Krylov. Let V_m satisfy $V_m^\top M V_m = I$ with first column $u^n / \|u^n\|_M$, and let $H_m = V_m^\top A V_m$ be the exact projection (as obtained from the Arnoldi relation [32, eq. (21)] for the polynomial method and from [32, eq. (24)] for the rational one). Then for every m and every u^n the Krylov step (13) satisfies $\|u^{n+1}\|_M \leq \|u^n\|_M$. The statement does not cover the approximation [32, eq. (25)] of H_m .

Proof. Let $L = -S$, so $\langle Lv, v \rangle_M \leq 0$. Let $0 < c_1 < \dots < c_m < 1$ and $b_i > 0$ be the Gauss–Legendre nodes and weights on $[0, 1]$ and $t_i = c_i \Delta t$. The collocation step seeks a polynomial y of degree $\leq m$ with values in $\mathbb{R}^N$ such that $y(0) = u^n$ and $y'(t_i) = Ly(t_i)$ for $i = 1, \dots, m$, and sets $u^{n+1} = y(\Delta t)$. Gauss quadrature with these nodes is exact for polynomials of degree $\leq 2m - 1$.

(a) *Uniqueness and existence.* The collocation conditions form a square linear system for the coefficients of $t, \dots, t^m$. Suppose $y(0) = 0$, so $y(t) = tz(t)$ with $\deg z \leq m - 1$. The polynomial $q(t) = \langle y'(t), z(t) \rangle_M$ has degree $\leq 2m - 2$, hence

$$\int_0^{\Delta t} q dt = \Delta t \sum_i b_i \langle Ly(t_i), y(t_i) \rangle_M / t_i \leq 0.$$

On the other hand $y' = z + tz'$ gives $q = \|z\|_M^2 + \frac{t}{2} \frac{d}{dt} \|z\|_M^2$, and integration by parts yields $\int_0^{\Delta t} q dt = \frac{1}{2} \int_0^{\Delta t} \|z\|_M^2 dt + \frac{\Delta t}{2} \|z(\Delta t)\|_M^2 \geq 0$. Thus $z \equiv 0$, the system has only the trivial solution for $u^n = 0$, and it is uniquely solvable for every u^n . *Contraction.* The polynomial $\langle y', y \rangle_M$ has degree $\leq 2m - 1$, so

$$\|y(\Delta t)\|_M^2 - \|y(0)\|_M^2 = 2 \int_0^{\Delta t} \langle y'(t), y(t) \rangle_M dt = 2\Delta t \sum_i b_i \langle Ly(t_i), y(t_i) \rangle_M \leq 0.$$

The m -stage Gauss method is the collocation method at the Gauss nodes; for $m = 1$ it is the midpoint rule, and its stability function is the (m, m) Padé approximant used in [32].

(b) For $u(t) = \exp(-tS)u^n$, $\frac{d}{dt} \|u\|_M^2 = -2\langle Su, u \rangle_M \leq 0$.

(c) Since $V_m^\top M V_m = I$, we have $\|V_m y\|_M = \|y\|_2$ for every $y \in \mathbb{R}^m$, so

$$\|u^{n+1}\|_M = \|u^n\|_M \|\exp(-\Delta t H_m) e_1\|_2.$$

The matrix $H_m = V_m^\top A V_m$ satisfies $H_m + H_m^\top = V_m^\top (A + A^\top) V_m \geq 0$, and (b) with $M = I$ gives $\|\exp(-\Delta t H_m)\|_2 \leq 1$. $\square$

In (c) the exact projection is essential: the proof uses only $H_m + H_m^\top = V_m^\top(A + A^\top)V_m \geq 0$, which an approximation of H_m such as [32, eq. (25)] need not inherit. Parts (a) and (b) are classical: the collocation argument is the one behind the B-stability of Gauss methods [5, 22], and (b) is the Lumer–Phillips estimate; the well-definedness of the implicit Gauss methods for dissipative operators is also noted in [32, Sect. 5]. We include the short proofs to keep the layer statements self-contained. What Theorem 6.1 adds is the reading of the integrators of [32] as one family of layers with a common guarantee, including the state-dependent Krylov layers.

6.2 Structure of the layers

State-dependent layers. The Krylov layer (13) is not a fixed matrix: V_m and H_m depend on u^n . It is positively homogeneous, $u^n \mapsto cu^n$ ($c > 0$) leaves V_m and H_m unchanged, but it is not additive (Section 7.5). In network language it is an input-adaptive layer whose weights are chosen by the algorithm.

Locality. Euler and the Runge–Kutta layers are exactly banded: a polynomial of degree s in S couples s neighbouring cells. The implicit and exponential layers are dense matrices, but numerically banded because waves travel a finite distance per step; Section 7.5 measures the band width. A large step is therefore a wide layer, and a sparse network pays either with many small steps or with a truncation of a dense layer.

6.3 Learned layers with built-in dissipativity

If the layer is to be learned from data, the structure of Theorem 6.1(a) can be imposed by construction. For any matrices P and Q put $B = (P - P^\top) + QQ^\top$, so that $B + B^\top = 2QQ^\top \geq 0$, and

$$W = (I + B)^{-1}(I - B). \quad (14)$$

Then $\|(I + B)u\|^2 - \|(I - B)u\|^2 = 2\langle (B + B^\top)u, u \rangle \geq 0$, so $\|W\|_2 \leq 1$ for every choice of the parameters. In energy coordinates every midpoint layer of a dissipative operator has this form (with $B = \frac{\Delta t}{2}\tilde{S}$), so this class contains the teacher of our experiment; Section 7.5 reports what this buys.

7 Experiments

7.1 Setup and reproducibility

All computations use double precision NumPy [23] for the operators, the checks and the constructive networks, and PyTorch [49] (single precision, CPU) for the trained advection–diffusion–reaction models. Every number in this section is written by the scripts in the accompanying code folder into the tables of the paper; a test suite of 285 checks, 172 of them for the lemmas and theorems of this paper, passes. The trained Runge–Kutta DG cells take between 11 and 134 minutes each, depending on ν and on the load of the machine; we reran seeds 42 and 44 for $\nu = 10^{-3}$ and $\nu = 10^{-4}$ and reused the stored output of the earlier run for the other cells, which are marked in Table 7. Everything else was rerun for this paper. Every rerun, including the trained baselines, the learned wave layers and the four retrained RK3–DG cells, reproduces the stored outputs of the earlier runs to the last digit (up to $2 \cdot 10^{-15}$ relative in two wave quantities).

Table 4: Largest $c = \beta\Delta t/h$ at which one step has norm ≤ 1 (two steps for “RK4²”), pure advection ($\nu = 0$), periodic, $N_e = 64$. “Euler” is the relaxed bound $\|I - \Delta t A\| \leq 1 + \Delta t$; the strict bound fails for every $c > 0$ when $r \geq 1$ (Proposition 4.1). A norm counts as ≤ 1 up to a tolerance of 10^{-12} . RK4 was not computed for $r = 0$.

r	Euler ($\leq 1 + \Delta t$)	SSP-RK2	SSP-RK3 block	RK4	RK4 ²
0	1.016	1.000	1.256		
1	3×10^{-3}	0.333	0.187	0.394	0.456
2	5×10^{-4}	5×10^{-4}	0.106	4×10^{-3}	0.164
3	2×10^{-4}	2×10^{-4}	0.068	2×10^{-3}	0.103

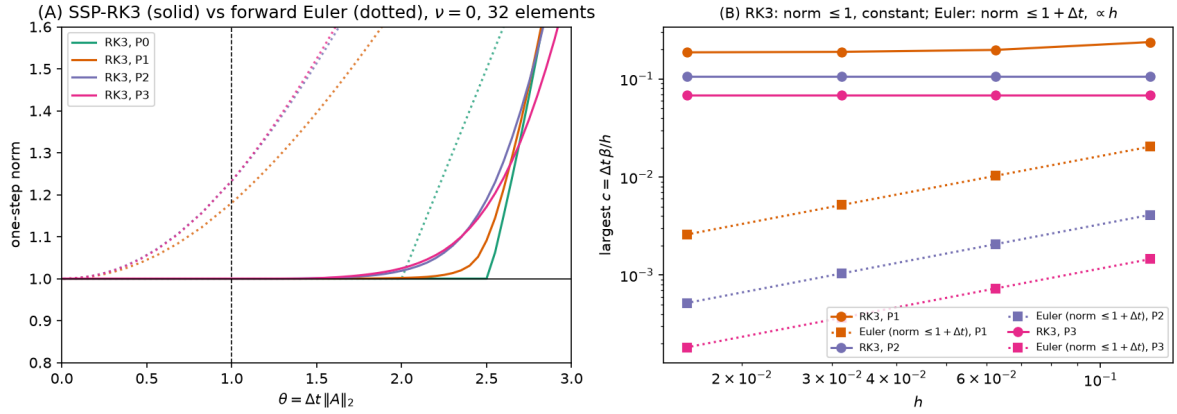


Figure 2: Left: $\|P_3(-\Delta t A)\|$ (solid) and the forward Euler norm (dotted) against $\theta = \Delta t\|A\|$, $\nu = 0$, periodic, $N_e = 32$. Right: largest admissible $c = \beta\Delta t/h$ against h , for the RK3 block (norm ≤ 1) and for forward Euler (relaxed bound $\leq 1 + \Delta t$).

7.2 Verification of the theory

We assembled the DG operator of Section 3.2 for $r = 0, \dots, 3$, $N_e \in \{8, 16, 32, 64\}$, $\nu \in \{0, 10^{-3}, 10^{-2}, 10^{-1}\}$ and both boundary conditions (128 configurations). The smallest eigenvalue of $A + A^\top$ is -1×10^{-11} , zero up to rounding, in agreement with Lemma 3.3. For $\theta = \Delta t\|A\| \leq 1$, the largest value of $\|P_3(-\Delta t A)\|$ is 1 plus 2×10^{-15} (rounding), as Theorem 5.1 predicts, and the identity (11) holds with a relative residual of at most 6×10^{-15} . The largest admissible θ^* (the first θ at which the norm exceeds 1) lies between 1.12 and 2.51, so the proved constant is conservative by at most this factor. Table 2 compares the constants of Proposition 5.3 with the measured norms, and Table 4 the step limits of several methods for pure advection. Figure 2 shows the block norm against θ and the admissible CFL number against h : constant for the RK3 block, proportional to h for the relaxed Euler bound $\|L_h\| \leq 1 + \Delta t$.

7.3 Constructive networks

Implicit versus explicit layers. Table 5 repeats the experiment of [62] for Fisher–KPP ($\nu = 1$, $T = 0.15$, $v_0(x) = \frac{1}{4} \sin(\pi x)(2 + \frac{1}{2} \sin 2\pi x - \frac{3}{10} \cos 2\pi x)$) with the constructive ReLU network. The explicit layer needs $\Delta t = 0.15h^2$; the IMEX–Euler layer (7) and the second-order IMEX layer (Crank–Nicolson for diffusion, Adams–Bashforth for the reaction; it carries two states) use $\Delta t \approx 0.2h$. IMEX–Euler is first order in time and therefore less accurate at the same mesh; the second-order IMEX layer matches the explicit accuracy with 170.7 times fewer layers at $h = 1/128$. Lemma 4.3 covers IMEX–Euler; for the two-step IMEX layer we prove no perturbation bound here.

Table 5: Constructive ReLU network for Fisher-KPP: number of layers N_t and L^2 error against a fine reference, for the explicit layer ($\Delta t = 0.15h^2$), IMEX-Euler and second-order IMEX ($\Delta t \approx 0.2h$). Deterministic. Ratio = explicit N_t / IMEX N_t .

N	h	explicit		IMEX-Euler		IMEX, 2nd order		ratio
		N_t	L^2 error	N_t	L^2 error	N_t	L^2 error	
7	1/8	64	4.62×10^{-3}	6	1.24×10^{-2}	6	3.47×10^{-3}	10.7
15	1/16	256	1.16×10^{-3}	12	7.30×10^{-3}	12	8.65×10^{-4}	21.3
31	1/32	1024	2.91×10^{-4}	24	3.93×10^{-3}	24	2.15×10^{-4}	42.7
63	1/64	4096	7.21×10^{-5}	48	2.04×10^{-3}	48	5.29×10^{-5}	85.3
127	1/128	16384	1.75×10^{-5}	96	1.04×10^{-3}	96	1.26×10^{-5}	170.7

Table 6: Constructive networks for (3), $N_e = 32$: continuous Galerkin with forward Euler (CG) against the RK3 DG block of degree $r = 1, 2$ at the proved step. Relative L^2 error in percent and TV ratio of the network output. Deterministic.

ν	Pe_h	CG, Euler		DG $r = 1$, RK3 block			DG $r = 2$, RK3 block		
		err. %	TV	N_t	err. %	TV	N_t	err. %	TV
10^{-1}	0.16	0.29	1.01	4831	0.33	1.00	20584	0.10	1.00
$2 \cdot 10^{-2}$	0.78	4.16	1.10	1051	2.09	0.98	4286	0.39	1.00
$5 \cdot 10^{-3}$	3.12	16.96	2.33	342	5.86	1.14	1231	2.71	1.13
10^{-3}	15.62	39.22	8.18	153	4.05	1.05	417	3.56	1.35
10^{-4}	156.25	73.73	21.31	111	0.58	1.06	235	0.63	1.11

Continuous Galerkin versus the RK3 DG block. Table 6 uses (3) with $\beta = 1$, $T = 0.55$, homogeneous Dirichlet conditions, $N_e = 32$, initial value $u_0 = \frac{1}{2} \sin(\pi x)(1.2 + 0.45 \sin 2\pi x - 0.35 \cos 2\pi x)$, and a reference solution on 2048 cells. The continuous Galerkin column uses centred $\mathbb{P}_1$ elements with forward Euler at 85% of the eigenvalue-based limit (capped at $0.25h$), as in the earlier draft. The DG columns use the plain upwind/interior-penalty operator of Section 3.2 with the RK3 block at the proved step $\Delta t = T/[T\|A\|_2]$ and the square replaced by a clipped Yarotsky network ($m = 6$ levels, $M_R = 2$). The TV ratio is the total variation of the output divided by that of the reference; values far above 1 indicate oscillations. In the advection-dominated rows the continuous Galerkin solution oscillates, as expected for centred discretisations at $Pe_h > 1$ [54] (TV ratio up to 21.3), while the TV ratio of the DG block stays at or below 1.35; at $\nu = 10^{-4}$ the $\mathbb{P}_1$ block needs 111 steps. In the diffusion-dominated rows the norm-based step is small, because $\|A\| \sim \nu/h^2$; this is where the implicit layers of Section 4.2 are the better choice. The comparison is thus between a proved, oscillation-free step and the number of layers it costs, not between two equally covered schemes. The network and the exact-square scheme differ by at most 0.01 percentage points, consistent with Lemma 5.5.

7.4 Trained operators for advection–diffusion–reaction

We learn the map $u_0 \mapsto u(\cdot, 0.55)$ for (3) with $\beta = 1$ and homogeneous Dirichlet conditions, from 120 training and 30 test initial values $u_0 = \frac{1}{2} \sin(\pi x)(1.2 + a_1 \sin 2\pi x + a_2 \cos 2\pi x)$, $a_{1,2} \sim U(-0.6, 0.6)$, sampled at 32 sensors, with the solution evaluated at 128 points and computed on 1024 cells. All models are trained with full-batch Adam [35] on the root-mean-square loss, learning rate $5 \cdot 10^{-3}$ with cosine decay to 10^{-4} , for 3500 epochs.

RK3-DG is the block (9) with the plain upwind/interior-penalty operator ($r = 1$, $N_e = 16$),

Table 7: Relative L^2 test error (%) of trained operators, mean $\pm$ standard deviation over the 30 test cases, one trained model per cell (seed 42; RK3-DG has 1–3 training seeds per regime and the baselines 5, see Table 8); parameter counts in parentheses; best per row in bold. † : stopped after 500 epochs (one epoch took about 12s at $\nu = 10^{-1}$ because the proved step needs 1 230 blocks); ‡ : stored output of the earlier run (not rerun, see Section 7.1).

ν	Pe	RK3-DG (321)	RK3-DG, exact square	NNO-disc (9 376)	FEM-NNO (9 376)	FERN (21 824)	DeepONet (75 660)
10^{-1}	10	$2.48 \pm 0.72^{\dagger\ddagger}$	1.20 ± 0.08	0.23 ± 0.01	0.27 ± 0.01	0.47 ± 0.03	0.09 ± 0.04
$2 \cdot 10^{-2}$	50	$5.61 \pm 0.67^\ddagger$	5.63 ± 0.68	0.77 ± 0.09	1.95 ± 0.14	1.78 ± 0.19	0.13 ± 0.06
$5 \cdot 10^{-3}$	200	$7.04 \pm 1.19^\ddagger$	7.04 ± 1.19	0.32 ± 0.07	5.64 ± 0.88	5.55 ± 0.87	0.33 ± 0.19
10^{-3}	1 000	2.98 ± 0.41	3.00 ± 0.41	0.76 ± 0.14	8.47 ± 1.48	8.42 ± 1.49	0.57 ± 0.35
10^{-4}	10 000	1.95 ± 0.76	1.94 ± 0.78	1.63 ± 0.15	8.56 ± 1.54	8.47 ± 1.55	0.78 ± 0.44

Table 8: Spread over training seeds: mean $\pm$ standard deviation of the per-seed test-mean error (%). Baselines: seeds 42–46 (all rerun). RK3-DG: number of seeds in the second column, using all available cells (rerun and stored); its spread is indicative only.

ν	RK3 seeds	RK3-DG	NNO-disc	FEM-NNO	FERN	DeepONet
10^{-1}	1	2.48 ± 0.00	0.56 ± 0.40	0.25 ± 0.05	0.35 ± 0.14	0.08 ± 0.01
$2 \cdot 10^{-2}$	1	5.61 ± 0.00	0.63 ± 0.19	1.88 ± 0.08	1.80 ± 0.05	0.12 ± 0.01
$5 \cdot 10^{-3}$	2	7.04 ± 0.00	0.59 ± 0.33	5.95 ± 0.71	5.78 ± 0.49	0.39 ± 0.09
10^{-3}	3	2.99 ± 0.01	0.45 ± 0.35	8.47 ± 0.05	8.43 ± 0.01	0.56 ± 0.09
10^{-4}	3	1.96 ± 0.01	0.46 ± 0.58	8.52 ± 0.06	8.48 ± 0.01	0.75 ± 0.11

the step $\Delta t = T/[T\|A\|_2]$ of Theorem 5.1, a fixed encoder (the L^2 projection of the piecewise linear interpolant of the sensor values) and a fixed decoder (the DG expansion, with $u(0) = u(1) = 0$ at the end points). Its only trained part is a scalar closure F that replaces the square in $u(1 - u)$: a $1 \rightarrow 16 \rightarrow 16 \rightarrow 1$ tanh network shared by all stages and steps and initialised to $F \equiv 0$, 321 parameters in total. Because the linear part is fixed and satisfies Theorem 5.1, the trained model inherits the stability of the block whatever F is learned (with C_{lip} replaced by the Lipschitz constant of F). *RK3-DG, exact square* is the same scheme with $F(s) = s^2$ and no training; it is the discretisation floor of the fixed operator and encoder. The baselines follow [62]: *FEM-NNO* is Zhang’s unrolled coefficient network (three blocks $v \mapsto A_\ell v + C_\ell \tanh(G_\ell v + b_\ell)$ with learned dense matrices) with a learnable hat basis; *NNO-disc* is the same coefficient network with a learnable discontinuous basis (it was called DG-NNO in an earlier draft, but it uses neither a DG operator nor a time-stepping scheme); *FERN* uses stacked branch networks with a learnable hat basis [63]; *DeepONet* uses stacked branch networks with a global tanh trunk [42].

Table 7 should be read as a comparison of stability against flexibility. RK3-DG fixes the mesh ($N_e = 16$), the encoder and the decoder, keeps the linear part of every step provably non-expanding, and learns only a scalar closure. The baselines learn their representation (a basis and a coefficient map) from the data; this buys accuracy on this one family of initial values but no stability guarantee. The trained closure reaches the error of the exact square wherever it was trained for the full 3500 epochs (at $\nu = 10^{-4}$: 1.95 % against 1.94 % for seed 42, and 1.96 ± 0.01 % over 3 seeds, Table 8), so training recovers the square and the remaining error is the discretisation floor of the fixed operator: $N_e = 16$, $\mathbb{P}_1$, a 32-sensor encoder and an unresolved boundary layer for small ν . Correspondingly its spread over seeds is small. With 321 parameters it is more accurate than the conforming models FEM-NNO and FERN for $\nu \leq 10^{-3}$, but *less* accurate than NNO-disc and DeepONet in every row, because those models

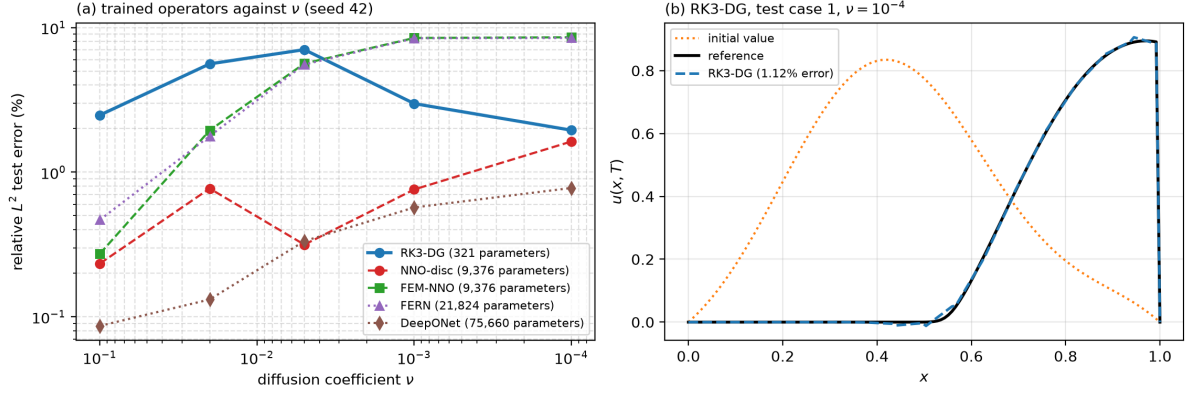


Figure 3: Trained operators across ν (seed 42; RK3-DG cells as in Table 7). Right: test case 1 at $\nu = 10^{-4}$ for RK3-DG.

Table 9: Energy norm $\|W\|_M$ of the wave layers against $\Delta t\rho(S)$; acoustics, $r = 2$, 16 cells. An entry 1 means equal to 1 within 10^{-12} ; $1 + x$ means larger than 1 by x .

$\Delta t\rho(S)$	Euler	RK3 block	RK4	midpoint	Gauss-3	exp
0.05	$1 + 5 \times 10^{-4}$	1	$1 + 2 \times 10^{-12}$	1	1	1
0.5	1.0592	1	$1 + 3 \times 10^{-7}$	1	1	1
1	1.2460	1	$1 + 1 \times 10^{-5}$	1	1	1
2	1.9048	1.0246	1.0010	1	1	1
2.8	2.5637	1.6763	1.0966	1	1	1
30	29.9	4×10^3	3×10^4	1	1	1

adapt their representation to the data while RK3-DG is limited by its fixed mesh and encoder. At $\nu = 10^{-1}$ the proved step makes training slow, and the run stopped at 500 epochs is not converged. This is the diffusion-dominated regime, in which the implicit layers of Section 4.2 are the better tool. The good results of NNO-disc are not covered by any lemma here: it is a black-box network. Figure 3 shows the errors across ν and one test case.

7.5 Wave layers

The wave experiments use homogeneous periodic acoustics ($\rho = c = 1$) with upwind DG of degree $r = 2$ on 16 cells ($N = 96$ unknowns, spectral radius $\rho(S) = 189.5$), unless stated otherwise; the dissipativity tests also use a layered medium ($\rho = (1, 2)$, $c = (1, 1.5)$, interface at $x = \frac{1}{2}$) with pressure-release boundaries.

Exact layers. Table 9 lists the energy norm of each layer of Table 3 for steps $\Delta t\rho(S)$ between 0.05 and 30. The midpoint, Gauss and exponential layers stay at 1 for every step, as Theorem 6.1 predicts; forward Euler exceeds 1 at every step; the RK3 block stays at 1 up to its limit, while RK4 exceeds 1 slightly even inside its stability region (consistent with [56, Prop. 3.1]), and both grow quickly beyond their stability limits. The layers reproduce the formulas of [32]: our midpoint layer agrees with $u - \Delta t(M + \frac{\Delta t}{2}A)^{-1}\Delta tAu$ to 4×10^{-15} , and the Gauss layer with the chain of complex resolvents to 6×10^{-14} . The polynomial Krylov layer has norm at most 0.82 on random states for $m \leq 30$, the rational one at most 0.78; it is homogeneous (defect 7×10^{-16}) but not additive (relative defect 0.17 at $m = 5$).

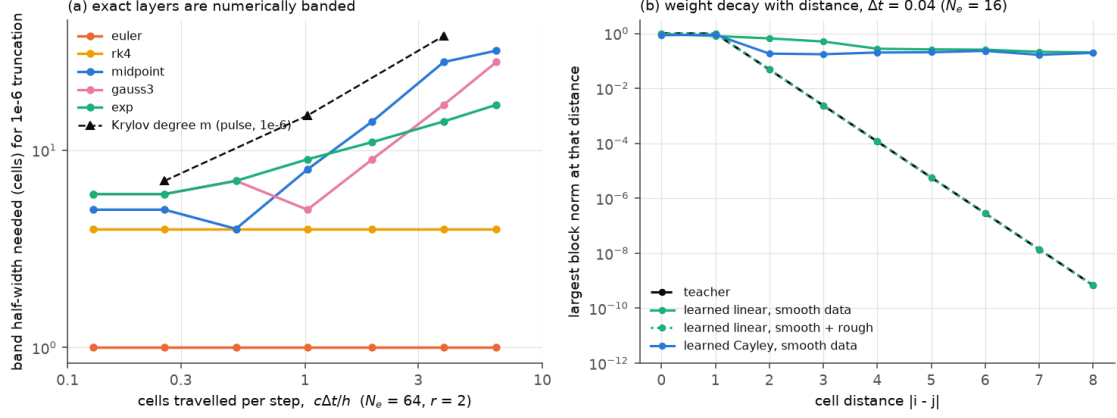


Figure 4: (a) Band half-width (cells) for a 10^{-6} truncation of each exact layer against the cells travelled per step, with the polynomial Krylov degree needed on a pulse (64 cells). (b) Largest weight block at each cell distance for the teacher (midpoint layer) and learned layers at $\Delta t = 0.04$ (16 cells): locality is learned only from data that excite rough states. Acoustics, $r = 2$.

Locality. On 64 cells, the half-width (in cells) of the band that keeps all neglected weight blocks below 10^{-6} grows with the distance travelled per step: from 5 to 32 cells for the midpoint layer, 6 to 28 for Gauss-3 and 6 to 17 for the exponential, as the wave moves 0.13 to 6.4 cells per step (Figure 4). The polynomial Krylov degree needed for a 10^{-6} step on a pulse tracks this width but exceeds it (7, 15, 38 at 0.26, 1.02, 3.84 cells per step, against bands of 6, 9, 14 cells).

The RK3 block against the implicit layers. At $\Delta t \|S\|_M = 1$ (the limit of Theorem 5.1) and 1000 steps, the RK3 block never increases the energy, like the midpoint, Gauss and exponential layers (Table 10); it damps rough states slightly more than the exact flow. Beyond its stability limit it blows up, while the implicit layers stay non-expanding at every step. At $T = 1$ the RK3 block is third-order accurate (Figure 5); the three-stage Gauss layer is far more accurate per step, at the price of a dense weight. Among the exact layers the trade-off is therefore sparsity and cost (explicit, stable up to a step limit) against unconditional stability and accuracy per step (implicit, dense).

Learned layers. We trained single-step layers to imitate the midpoint layer (the teacher) at $\Delta t \in \{0.01, 0.02, 0.04, 0.08, 0.16\}$ ($\Delta t \rho(S)$ from 1.9 to 30), from one-step pairs of 120 smooth pulses (“smooth”) or of 120 smooth pulses and 30 rough random states (“mixed”), eight teacher steps each. The students are a linear layer W (initialised at the identity), a layer $Wx + C \text{ReLU}(Gx + b)$ of the form (1) with 64 hidden units, and the Cayley layer (14); all are trained with Adam and evaluated by 1000-step rollouts on 20 unseen pulses, three seeds each. Table 11 shows the medians at $\Delta t = 0.04$ and $\Delta t = 0.16$. Three observations, from the good to the bad:

- With rough states in the data, the linear student recovers the teacher to rounding error at every step size (one-step error $\leq 3 \times 10^{-16}$, $\leq 3 \times 10^{-14}$ after 1000 steps): the weights become the discrete operator.
- The Cayley student is stable in every run (peak energy exactly 1) and accurate to 7×10^{-4} – 2×10^{-2} after 1000 steps, even from smooth data. Its class contains the teacher, so this shows that the structure is learnable, not that it generalises beyond linear dissipative problems.
- Unconstrained students trained on smooth data alone reach one-step errors of 1×10^{-3} to 2×10^{-2} and still blow up within 1000 steps; their weights have norms 1.4 to 3.5, while

Table 10: 1000-step rollouts (20 smooth pulses and 20 rough states, median of the energy ratio E_{1000}/E_0) and accuracy at $T = 1$ for smooth pulses against $\exp(-TS)$; acoustics, $r = 2$, 16 cells. $c = \Delta t \|S\|_M$.

layer	c	$\ W\ _M$	peak energy	final, smooth	final, rough
RK3 block	0.5	1	1	0.9917	0.2422
RK3 block	1	1	1	0.9820	0.2100
RK3 block	1.25	1	1	0.9756	0.1984
RK3 block	3	1.9220	5×10^{121}	blow-up	blow-up
forward Euler	1	1.2326	2×10^{25}	blow-up	blow-up
RK4	1	$1 + 1 \times 10^{-5}$	1	0.9841	0.2133
midpoint	1	1	1	0.9841	0.2136
Gauss-3	1	1	1	0.9841	0.2133
exp (exact)	1	1	1	0.9841	0.2133

Error at $T = 1$, smooth pulses, $c \approx 1$ (195 steps):

RK3 4.8×10^{-4} , RK4 1.3×10^{-5} , midpoint 8.4×10^{-3} , Gauss-3 4.4×10^{-10} , Euler 1.74

the teacher has norm 1. Smooth data never excite many directions, and an unconstrained layer is free to amplify them. The ReLU part does not help on this linear problem: with mixed data it blows up at $\Delta t = 0.01$ and 0.02 (median over seeds), and for $\Delta t \geq 0.04$ its 1000-step error is 0.16 to 0.36.

The controls show that the outcome depends on the optimiser: plain gradient descent from the identity kept $\|W\| \leq 1.009$, and its rollouts did not blow up at any step size (peak energy at most 1.002, 1000-step errors of 0.05 to 0.11). Instability of learned layers is therefore not a law, but nothing in an unconstrained layer prevents it, whereas the Cayley layer excludes it by construction (Figure 6). This is the same stability-versus-flexibility trade-off as in Section 7.4: the unconstrained layers can represent any matrix and fit one step well; the Cayley layer can represent only non-expanding maps and is stable for every value of its parameters.

8 Limitations

- *Dimension and meshes.* All experiments are one-dimensional, on uniform meshes, with one linear and one quadratic model problem. Theorems 5.1 and 6.1 hold for any dissipative operator, but the constants of Proposition 5.3 are for uniform one-dimensional meshes; on graded meshes the step limit is set by the smallest cell.
- *Stability is not accuracy.* A non-expanding layer can still be biased: the RK3 block dissipates rough modes more than the exact flow, and the trained RK3-DG model is limited by its mesh.
- *Training.* The trained models use one data set (120/30 samples), no hyperparameter search, and few seeds for RK3-DG; the cell at $\nu = 10^{-1}$ is not converged, and several RK3-DG cells are stored outputs of an earlier run rather than reruns.
- *Learned wave layers.* One linear, homogeneous, periodic problem on one mesh, three seeds; the teacher lies in the Cayley class.
- *Proofs not given.* No perturbation bound for the two-step IMEX layer of Table 5; the network-size statement (Corollary 4.4) is conditional on a discretisation estimate that we did not verify for the semilinear problem; no complexity theorem for the RK3 block.

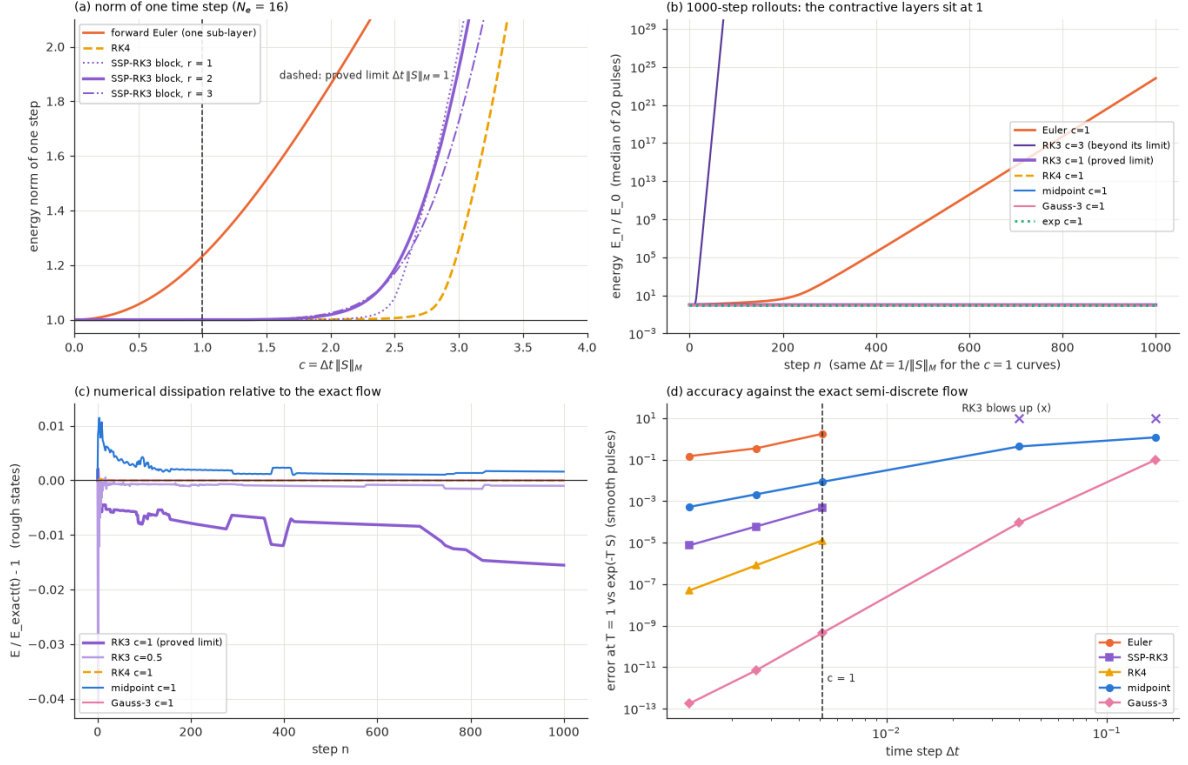


Figure 5: RK3 block on the wave problem: layer norm against $c = \Delta t \|S\|_M$; energy in 1000-step rollouts; dissipation of rough states against the exact flow; error at $T = 1$ against the number of steps.

9 Outlook and extensions

The results above use two ingredients only: a dissipative operator A (Definition 3.1) and a time integrator that inherits this property. For a new equation, mesh or splitting, the question is therefore whether the resulting layer is a composition of non-expanding pieces. We outline five directions in which this carries over, saying what is covered by the theory above and what is open; none of them is tested in this paper.

Maxwell's equations and conducting media. The paper [32] derives the upwind flux for Maxwell's equations and the face-dissipation identity for perfectly conducting boundaries [32, p. 247] (the boundary jump of Remark 3.4). With $M = \text{diag}(\varepsilon, \mu)$ the operator is dissipative in the electromagnetic energy norm, so Theorems 5.1 and 6.1 apply as stated. A conductivity $\sigma \geq 0$ adds $\int \sigma |E|^2 \geq 0$ to (Au, u) and keeps dissipativity, but $\|S\|_M$ then grows at least like σ/ε , so the condition $\Delta t \|S\|_M \leq 1$ of Theorem 5.1 forces small steps, while the implicit and exponential layers are unaffected. Open are the constants of Proposition 5.3 in two and three space dimensions, interfaces and absorbing boundaries, and the cost of dense implicit layers against banded explicit ones. Implicit Runge–Kutta methods for DG Maxwell are analysed in [30], and energy stability of DG time-domain methods on heterogeneous media in [17].

Learned materials. For any positive ε, μ and $\sigma \geq 0$ the Maxwell operator is dissipative in its own energy norm. If the coefficients are parametrised to stay positive, the layer is therefore non-expanding for every parameter value, in the spirit of the Cayley layer of Section 6.3 but with the structure fixed by the physics rather than learned. The energy norm depends on the parameters, so comparing layers in a fixed norm costs a factor that depends on the material contrast; bounding it is open, as is whether the guarantee is useful for learning interface posi-

Table 11: Learned wave layers (teacher: midpoint layer), median over three seeds. Errors are relative to the teacher on unseen smooth pulses; “peak energy” is the largest E_n/E_0 over 1000 steps; $\|W\|$ is the spectral norm of the linear part in energy coordinates.

data	student	Δt	1-step error	peak energy	$\ W\ $	1000-step error
smooth	linear	0.04	2.3×10^{-3}	9×10^{121}	2.4947	blow-up
smooth	linear	0.16	2.8×10^{-3}	1×10^{122}	3.5189	blow-up
smooth	ReLU	0.04	1.6×10^{-2}	9×10^{121}	2.0541	blow-up
smooth	ReLU	0.16	2.0×10^{-2}	8×10^{121}	2.2937	blow-up
smooth	Cayley	0.04	1.7×10^{-3}	1	1.0000	7.3×10^{-3}
smooth	Cayley	0.16	1.1×10^{-3}	1	1.0000	1.7×10^{-2}
mixed	linear	0.04	2.5×10^{-16}	1	1	2.1×10^{-14}
mixed	linear	0.16	3.5×10^{-16}	1	1	3.3×10^{-14}
mixed	ReLU	0.04	1.7×10^{-2}	1.0201	2.4864	0.36
mixed	ReLU	0.16	2.4×10^{-2}	$1 + 3 \times 10^{-4}$	3.2635	0.17
mixed	Cayley	0.04	2.1×10^{-6}	1	1.0000	6.9×10^{-4}
mixed	Cayley	0.16	9.5×10^{-4}	1	1.0000	2.2×10^{-2}

tions. That wave physics can be read as a recurrent network whose programmable part is the material is shown in [34].

Local time stepping and multirate layers. A locally refined mesh makes $\|S\|_M$ large, and local time stepping replaces one layer with a product of block-sparse sub-layers, with a coarse step. A product of non-expanding layers is non-expanding, so the question is whether the pieces are; a split of A into a coarse and a fine part need not give two dissipative parts. Locally implicit integration for upwind DG discretisations of Maxwell’s equations is analysed in [31], explicit local time stepping for Maxwell’s equations in [19]. Open is a split with dissipative parts whose step depends on the coarse cells only, and its sparsity and parameter count as a network.

Learned contractive rational and Krylov layers. A product of midpoint layers with different positive step sizes is non-expanding, since each factor is covered by Theorem 6.1(a); the step sizes can thus be treated as parameters. The open question is how close a product with few factors, that is, few linear solves, comes to $\exp(-\Delta t S)$ on the spectrum of a DG operator, compared with Gauss collocation and rational Krylov methods with tuned shifts. The choice of shift and step is called nontrivial in [32]; optimal pole selection is studied in [20].

Nonlinear conservation laws. For nonlinear fluxes the proofs above do not apply: they use linearity, and an entropy bound on one solution is not a bound on differences of solutions. What carries over are the ReLU pieces of Section 4.3 and the known L^2 stability of third-order Runge–Kutta DG for scalar conservation laws [61]. Entropy stable DG methods [10] and relaxation Runge–Kutta methods, which adapt the step by a scalar so that a discrete entropy is conserved or dissipated [50], suggest nonlinear layers whose scalar correction is part of the network; neural networks have been trained as limiter indicators in [51]. Open is a perturbation bound for such a layer, in the spirit of Lemma 5.5.

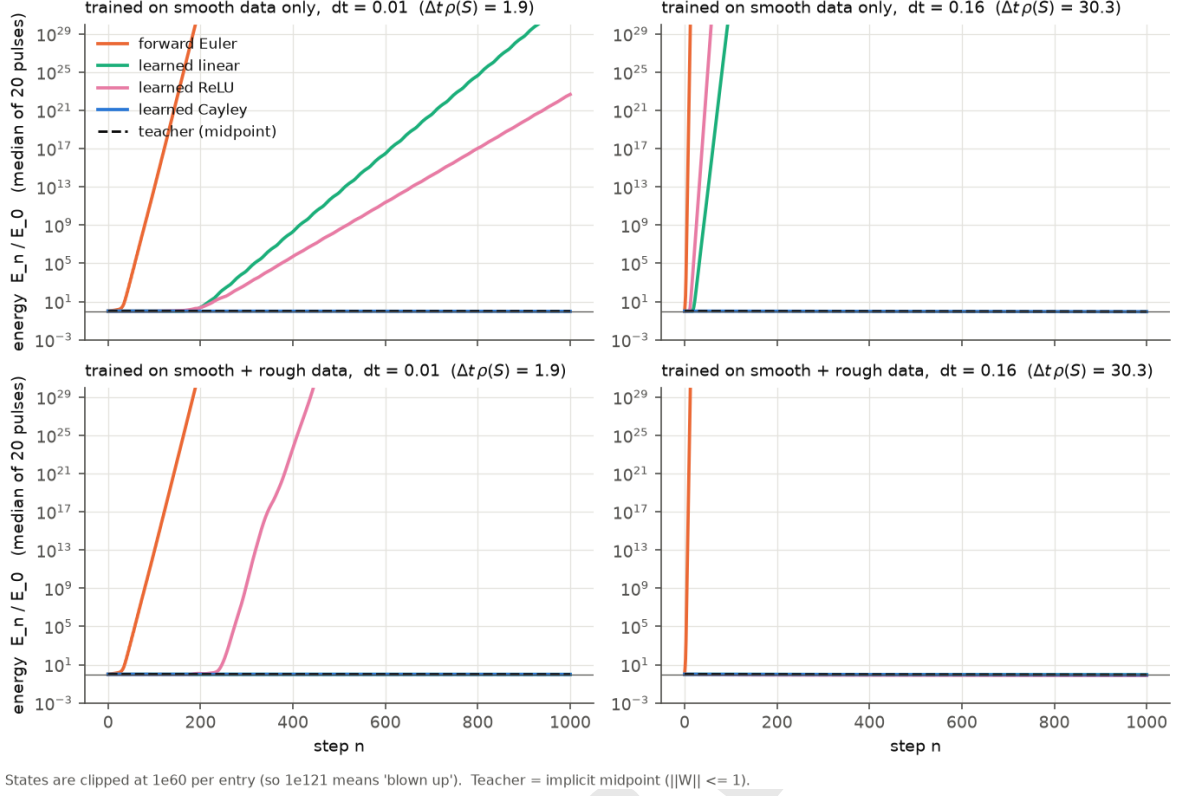


Figure 6: Energy in 1000-step rollouts (median of 20 unseen pulses) of the learned layers, trained on smooth data (top) and on mixed data (bottom), at $\Delta t = 0.01$ (left) and $\Delta t = 0.16$ (right), against forward Euler and the teacher. States are clipped at 10^{60} per entry, so energies near 10^{121} mean blow-up.

10 Conclusion

A time-stepping layer built from an upwind DG operator is safe to stack when the integrator inherits the operator's dissipativity. Forward Euler never does for degree $r \geq 1$; the SSP-RK3 block does under the hyperbolic condition $\Delta t \|S\|_M \leq 1$, with explicit constants; implicit, exponential and Krylov layers do for every step. The integrators of [32] thus form a family of non-expanding layers, and the same dissipativity can be imposed on learned layers by a Cayley parametrisation. The experiments confirm the bounds and show the trade-off between stability and flexibility: a model with a fixed mesh and encoder and a provably stable linear part (321 parameters, $1.96 \pm 0.01\%$ error over 3 seeds at $\nu = 10^{-4}$) does not reach the accuracy of black-box operator networks that learn their representation, which in turn come with no stability guarantee. Extensions to Maxwell's equations, multirate layers, learned rational layers and nonlinear fluxes are outlined in Section 9.

Reproducibility statement

The code folder that accompanies this paper contains the scripts that produce every table and figure, the test suite, and a single command (`code/rerun_all.sh`) that reruns everything except the long trained cells, which can be rerun one by one. Seeds are fixed throughout. `BUILD.md` lists which outputs were rerun and which were reused, with run times.

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